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Lie point symmetries for reduced Ermakov systems

F. Haas *, J. Goedert

Ciências Exatas e Tecnológicas, UNISINOS, Av. Unisinos, 950, 93022-000 São Leopoldo, RS, Brazil

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Abstract

The condition for Lie point symmetries for reduced Ermakov systems is solved yielding three families of systems. $SL(2, R)$ is always a group of symmetries when frequencies depends on time only. However, the generator of symmetries in more general cases have a contribution not associated with the $SL(2, R)$ group.

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1. Introduction

Ermakov systems [1–3] play an important role in a variety of physical and mathematical situations. The most recent analysis involving Ermakov systems deals with Bose–Einstein condensates and cosmological models [4–6], nonlinear supersymmetric Darboux transformations [7], the free fall of a quantum bouncing ball [8], conformal quantum mechanics [9], general covariance and time-dependent metrics in quantum mechanics [10], geometric phases [11] and generalized Hamiltonian structures [12]. From the theoretical viewpoint, Ermakov systems always admit a constant of motion, the Ermakov invariant, and are amenable to a nonlinear superposition law [13]. In addition, Ermakov systems are linearizable under broad circumstances [14,15].

As is well known, the group theoretic approach to a dynamical system is a subject of relevance [16] not only for the reduction of order and the search for invariants for the system, but also for a better understanding of its structural properties. The point symmetry group of Ermakov systems has been identified as the $SL(2, R)$ group in the case of frequency functions depending on time only and also for a large class of more general frequency functions [17–21]. More recently, using the converse to Noether's theorem, it has been shown that the Ermakov invariant can be asso-

* Corresponding author.
E-mail addresses: ferhaas@exatas.unisinos.br (F. Haas), goedert@exatas.unisinos.br (J. Goedert).

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ciated to a dynamical symmetry, in the cases where the system admits a variational formulation [22]. In addition, $SL(2,R)$ has also been found [23,24] as the symmetry group of Kepler–Ermakov systems, which can be viewed either as perturbations of the planar Kepler problem or of the classical Ermakov system. The purpose of this Letter is to follow this trend from a different perspective and study the Lie point symmetries of Ermakov systems restricted to manifolds where the Ermakov invariant has a fixed constant value. The importance of this study may not be underestimated since the existence of the Ermakov invariant is automatic. This point will be illustrated with an interesting example at the end of this section.

In polar coordinates, the Ermakov system reads

$$\ddot{r} - r\dot{\theta}^2 + \omega^2 r = \frac{F(\theta)}{r^3}, \tag{1}$$

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = \frac{G(\theta)}{r^3}, \tag{2}$$

where F and G are arbitrary functions of the angle θ and ω, in principle, can depend arbitrarily on the dynamical variables. More often, ω is a function of time only, in which case it has the interpretation of a time-dependent frequency. Here, however, we do not impose this constraint and allow for more general functional dependences. Also, for simplicity we consider the case $\partial\omega/\partial\dot{r} = 0$. Independently of the special form of ω, the Ermakov systems always possess the constant of motion

$$I = \frac{1}{2}\left(r^2\dot{\theta}\right)^2 - \int^{\theta} G(\phi)\,d\phi, \tag{3}$$

the so-called Ermakov invariant [1–3].

The existence of a constant of motion reduces the order of the system. More exactly, Eq. (2) can be integrated once to give

$$r^2\dot{\theta} = H(\theta), \tag{4}$$

where

$$H(\theta) = \sqrt{2}\left(I + \int^{\theta} G(\phi)\,d\phi\right)^{1/2}. \tag{5}$$

The structure of Eq. (4) suggests the introduction of the new variable

$$\varphi = \int^{\theta} \frac{d\phi}{H(\phi)}, \tag{6}$$

so that the fourth-order non-reduced Ermakov system (1)–(2) can always be cast in the reduced form

$$\ddot{r} = A(r,\varphi,t), \tag{7}$$

$$r^2\dot{\varphi} = 1, \tag{8}$$

where the function A is defined by

$$A(r,\varphi,t) = -\omega^2 r + \frac{1}{r^3}\left(F + H^2\right). \tag{9}$$

In (9), F and H are functions of φ through (6) and the implicit function theorem. The fact that $\partial\omega/\partial\dot{r} = 0$ ensures the indicated functional dependence of A. Notice, however, that ω can freely depend, for instance, on $\dot{\varphi}$, since this dependence can be eliminated through (8). The choice $\partial\omega/\partial\dot{r} = 0$ has a decisive influence on the simplification of the symmetry analysis but imposes some limitations on the generality of our results. In addition, since the symmetry analysis is insensitive to coordinate transformations, working with φ and not with θ does not affect the final result.

Eqs. (7)–(8) are a third-order dynamical system, which we shall call the *reduced Ermakov system*. Even if not explicitly shown, the reduced equations depend parametrically on I. Also, the function H is not identically zero except for the trivial case $I = G = 0$, which we do not consider here.

The purpose of this work is to perform the Lie point symmetry analysis of reduced Ermakov systems. Since for Lagrangian Ermakov systems the Ermakov invariant is directly related to a dynamical Noether symmetry [22], it is to be expected that the algebra $sl(2,R)$ will play a fundamental role on the reduced Ermakov system. Let us stress, however, that the reduction process may change significantly the symmetry analysis.

In order to illustrate the differences between the symmetry analysis of non-reduced and reduced systems, consider the third-order ordinary differential equation introduced in [25],

$$\dddot{q} + \frac{a\dot{q}\ddot{q}}{q} + \frac{b\dot{q}^3}{q^2} = 0, \tag{10}$$

where a and b are arbitrary constants. Eq. (10) admits the symmetry generators

$$U_1 = q\frac{\partial}{\partial q}, \qquad U_2 = t\frac{\partial}{\partial t} + 3q\frac{\partial}{\partial q}. \tag{11}$$

Moreover, Eq. (10) admits the invariant

$$I=\frac{\ddot{q}}{q^{2k+1}}-\frac{b}{2(k+1)}\frac{\dot{q}^2}{q^{2k+2}}, \tag{12}$$

where k is any root for

$$2k^2+(a+3)k+a+b+1=0. \tag{13}$$

Using this invariant, we obtain the reduced system

$$\ddot{q}-\frac{b}{2(k+1)}\frac{\dot{q}^2}{q}-Iq^{2k+1}=0. \tag{14}$$

For arbitrary I, the reduced system admits the symmetry generator U_1 only for $k=0$ (implying $a+b+1=0$). In a similar way, U_2 is admitted only if $k=-1/3$ (implying $a+3b/2+1/3=0$). Both symmetries are admitted only for $a=-4/3$, $b=1/3$. Therefore, the reduced system has symmetry properties bearing no resemblance with the symmetry properties of the original, non-reduced system. The same argument may apply to Ermakov systems implying that reduced Ermakov systems require their own symmetry analysis.

The organization of the Letter is as follows. In Section 2, the general symmetry conditions to be satisfied by reduced Ermakov systems and their symmetry generators are determined. In Section 3, we solve the symmetry conditions in the case of transformations of the time not involving the dynamical coordinates. This yields three classes of reduced systems admitting Lie point symmetries. In Section 4, we show that, for frequency functions depending on time only, the reduced Ermakov systems always admit $SL(2,R)$ as a symmetry group. This shows that, more properly, $SL(2,R)$ is the natural group of symmetries for *reduced* Ermakov systems, the reduction being possible as a consequence of a dynamical Noether symmetry in the case of Lagrangian Ermakov systems. In Section 5, we start from the equations for an ion under a generalized Paul trap and find the circumstances under which these equations, viewed as a reduced Ermakov system, do possess Lie point symmetries. Section 6 is devoted to the conclusions.

2. Lie symmetries

Consider infinitesimal point transformations of the form

$$\bar{r}=r+\varepsilon R(r,\varphi,t),$$
$$\bar{\varphi}=\varphi+\varepsilon S(r,\varphi,t),$$
$$\bar{t}=t+\varepsilon T(r,\varphi,t), \tag{15}$$

for functions R, S and T to be determined and infinitesimal parameter ε. The procedure for computing Lie symmetries is well known [16] and we limit ourselves to sketch the critical steps in our case. The above infinitesimal transformation will be a Lie point symmetry of the reduced Ermakov system if and only if (7)–(8) remains formally invariant under (15) up to first order in ε, in the solution set of the reduced Ermakov system. This symmetry condition will imply the vanishing of two separate polynomials in $\dot{r}$, one associated with the radial equation (7), the other associated with the angular equation (8). Imposing the vanishing of the coefficient of all different powers of $\dot{r}$ we get the following set of linear, coupled partial differential equations,

$$\frac{\partial^2 T}{\partial r^2}=0, \tag{16}$$

$$\frac{\partial^2 R}{\partial r^2}-\frac{2}{r^2}\frac{\partial^2 T}{\partial r\partial\varphi}-2\frac{\partial^2 T}{\partial r\partial t}+\frac{2}{r^3}\frac{\partial T}{\partial\varphi}=0, \tag{17}$$

$$\frac{2}{r^2}\frac{\partial^2 R}{\partial r\partial\varphi}+2\frac{\partial^2 R}{\partial r\partial t}-\frac{2}{r^3}\frac{\partial R}{\partial\varphi}-3A\frac{\partial T}{\partial r}-\frac{1}{r^4}\frac{\partial^2 T}{\partial\varphi^2}-\frac{2}{r^2}\frac{\partial^2 T}{\partial\varphi\partial t}-\frac{\partial^2 T}{\partial t^2}=0, \tag{18}$$

$$r^2\frac{\partial S}{\partial r}-\frac{\partial T}{\partial r}=0, \tag{19}$$

$$\frac{\partial S}{\partial\varphi}+r^2\frac{\partial S}{\partial t}-\frac{1}{r^2}\frac{\partial T}{\partial\varphi}-\frac{\partial T}{\partial t}+\frac{2R}{r}=0, \tag{20}$$

$$UA=\left(\frac{\partial R}{\partial r}-\frac{2}{r^2}\frac{\partial T}{\partial\varphi}-2\frac{\partial T}{\partial t}\right)A+\frac{1}{r^4}\frac{\partial^2 R}{\partial\varphi^2}+\frac{2}{r^2}\frac{\partial^2 R}{\partial\varphi\partial t}+\frac{\partial^2 R}{\partial t^2}. \tag{21}$$

In Eq. (21), U is the generator of symmetries,

$$U=R\frac{\partial}{\partial r}+S\frac{\partial}{\partial\varphi}+T\frac{\partial}{\partial t}. \tag{22}$$

The solutions of Eqs. (16)–(21) determine the Lie point symmetries of the reduced Ermakov system together with the classes of admissible functions A. In the following section we show three categories of solutions for the determining equations.

3. Exact solutions

A closer examination of (18) shows that the function A which specifies the dynamics of the radial variable in the reduced Ermakov system, will soon become, to some extent, determined if $\partial T/\partial r \neq 0$. Indeed, (16) and (17) immediately give the r dependence of R and T, which, in turn, will determine the r dependence of A through (18). Furthermore, using (19)–(20) shows that, for $\partial T/\partial r \neq 0$, A contains only two terms, one proportional to r, the other to r^{-3}. To avoid this excessively constrained situation we choose

$$\frac{\partial T}{\partial r} = 0, \tag{23}$$

a condition that will be assumed throughout this Letter.

Assuming this constraint on T, the solution for (16)–(20) is

$$R = \left(\rho(t)\dot{\rho}(t) - \frac{S'(\varphi)}{2}\right) r,$$
$$S = S(\varphi), \qquad T = \rho^2(t), \tag{24}$$

where ρ is an arbitrary function of time and S is an arbitrary function of φ. In (24) and in the sequel, a prime denotes derivation with respect to φ.

Until now, no constraint was imposed on the function A of the reduced Ermakov system, but there still remains the symmetry condition (21). Inserting (24) into (21), we get the following determining equation,

$$UA = -\left(3\rho\dot{\rho} + \frac{S'}{2}\right) A + (\rho\dddot{\rho} + 3\dot{\rho}\ddot{\rho}) r - \frac{S'''}{2r^3}, \tag{25}$$

where U is the generator of Lie point symmetries for reduced Ermakov systems,

$$U = \left(\rho\dot{\rho} - \frac{S'}{2}\right) r \frac{\partial}{\partial r} + S\frac{\partial}{\partial \varphi} + \rho^2 \frac{\partial}{\partial t}. \tag{26}$$

The vector field U contains two arbitrary functions, $\rho(t)$ and $S(\varphi)$.

The $sl(2,R)$ algebra is obtained from (26) in the particular case

$$\rho^2 = c_0 + c_1 t + c_2 t^2, \qquad S = 0, \tag{27}$$

where c_0, c_1 and c_2 are arbitrary numerical constants. The three generators of the $sl(2,R)$ algebra are obtained by taking separately each of these constants non-zero. In comparison with the generator of point symmetries of the non-reduced Ermakov system [17–21], the new ingredient of U is $S(\varphi)$. Also, $\rho^2(t)$ is not necessarily a second-degree polynomial in t. In Section 4, we present a more detailed account of the relation between the point symmetries of reduced Ermakov systems and the $sl(2,R)$ algebra.

Eq. (25) can be viewed either as an equation for A or for S. We feel more productive to consider it as the determining equation for A, since S participates in the generator through the definition (26). Following this choice, we find three classes of solutions for A, listed below. All these solutions are build using the differential invariants of the operator U, that is, the independent functions I_1 and I_2 for which $UI_1 = UI_2 = 0$.

3.1. The $\rho \neq 0$, $S \neq 0$ case

In this situation, the method of characteristics yields the following differential invariants for the generator U,

$$I_1 = \int^{\varphi} \frac{d\phi}{S(\phi)} - \int^{t} \frac{d\tau}{\rho^2(\tau)}, \tag{28}$$

$$I_2 = \frac{r}{\rho} \exp\left(\frac{1}{2}\int^{t} \frac{d\tau}{\rho^2(\tau)} S'\big(\varphi(\tau; I_1)\big)\right). \tag{29}$$

In (29), $\varphi = \varphi(t; I_1)$ is a function of t as given locally by the implicit function theorem through (28). The differential invariants can be used to construct the solution for (25). The result is

$$\begin{aligned} A = {} & \frac{\ddot{\rho}}{\rho} r - \frac{1}{2r^3} \exp\left(-2\int^{t} \frac{S'(\tau)}{\rho^2(\tau)} d\tau\right) \\ & \times \int^{t} \frac{d\mu}{\rho^2(\mu)} S'''(\mu) \exp\left(2\int^{\mu} \frac{S'(\nu)}{\rho^2(\nu)} d\nu\right) \\ & + \frac{1}{\rho^3} \exp\left(-\frac{1}{2}\int^{t} \frac{S'(\tau)}{\rho^2(\tau)} d\tau\right) \tilde{A}(I_1, I_2), \end{aligned} \tag{30}$$

where φ, in the integrals, is taken as a function of t through (28) and the implicit function theorem. $\tilde{A}$ is an arbitrary function of the differential invariants I_1 and I_2.

To summarize, the reduced Ermakov system (7)–(8) has a Lie point symmetry with generator (26) for $\rho \neq 0$ and $S \neq 0$, provided A can be cast in the form (30), including the arbitrary functions ρ and S. Notice that the

Ermakov invariant enters as a parameter in the symmetry generator as well as in the function A. This is no surprise since the Ermakov invariant was used to eliminate $\dot{\varphi}$ from the equations of motion.

The global transformations can be found from the infinitesimal transformations following the traditional procedure [16]. The result is

$$\bar{r} = \frac{\rho(\bar{t})}{\rho(t)} \exp\left(-\frac{1}{2}\int_t^{\bar{t}} \frac{S'(\tau)}{\rho^2(\tau)} d\tau\right) r,$$
$$\int_\varphi^{\bar{\varphi}} \frac{d\phi}{S(\phi)} = \epsilon, \qquad \int_t^{\bar{t}} \frac{d\tau}{\rho^2(\tau)} = \epsilon, \tag{31}$$

where now ϵ is a finite parameter.

3.2. The $\rho \neq 0$, $S = 0$ case

This case corresponds to the usual quasi-invariance transformations [17]. Now the differential invariants for U are

$$I_1 = \frac{r}{\rho}, \qquad I_2 = \varphi, \tag{32}$$

and the corresponding solution for (25) is

$$A = \frac{\ddot{\rho}}{\rho} r + \frac{1}{r^3} \tilde{A}\left(\frac{r}{\rho}, \varphi\right), \tag{33}$$

where $\tilde{A}$ is an arbitrary function of the indicated arguments. This class of solutions contains the arbitrary functions ρ and $\tilde{A}$, subject to $\rho \neq 0$. The global transformation is given by

$$\bar{r} = \frac{\rho(\bar{t})}{\rho(t)} r, \qquad \bar{\varphi} = \varphi, \qquad \int_t^{\bar{t}} \frac{d\tau}{\rho^2(\tau)} = \epsilon, \tag{34}$$

where ϵ is the finite parameter of the transformation.

3.3. The $\rho = 0$, $S \neq 0$ case

The differential invariants are

$$I_1 = S r^2, \qquad I_2 = t, \tag{35}$$

while

$$A = -\frac{1}{2r^3}\left(\frac{S''}{S} - \frac{S'^2}{2S^2}\right) + r\tilde{A}(I_1, I_2), \tag{36}$$

for $\tilde{A}$ an arbitrary function depending on the differential invariants of the symmetry generators. Now there are the free functions S and $\tilde{A}$. The global transformation is determined by

$$\bar{r} = \left(\frac{S(\varphi)}{S(\bar{\varphi})}\right)^{1/2} r, \qquad \int_\varphi^{\bar{\varphi}} \frac{d\phi}{S(\phi)} = \epsilon, \qquad \bar{t} = t, \tag{37}$$

where ϵ is the finite parameter of the transformation.

4. Connection with the $SL(2, R)$ group

$SL(2, R)$ is the Lie point symmetry group for nonreduced Ermakov systems with frequency functions depending on time only and also for some classes of more general frequency functions [17–21]. For the sake of comparison, we have to investigate the role of this transformations group for the reduced Ermakov systems. For simplicity, in this section we consider Ermakov systems containing frequencies depending on time only. In this case (see Eq. (9)) A is given by

$$A(r, \varphi, t) = -\Omega^2(t) r + \frac{1}{r^3}\Gamma(\varphi), \tag{38}$$

for a time-dependent frequency $\Omega(t)$ and where Γ is defined by

$$\Gamma(\varphi) = F(\theta) + H^2(\theta). \tag{39}$$

Inserting A in the symmetry condition (25), we get

$$\left(\rho\dddot{\rho} + 3\dot{\rho}\ddot{\rho} + 4\Omega^2\rho\dot{\rho} + 2\Omega\dot{\Omega}\rho^2\right) r$$
$$- \frac{1}{2}(S''' + 4\Gamma S' + 2\Gamma' S) r^{-3} = 0. \tag{40}$$

This equation has to be satisfied for arbitrary r and therefore

$$\rho\dddot{\rho} + 3\dot{\rho}\ddot{\rho} + 4\Omega^2\rho\dot{\rho} + 2\Omega\dot{\Omega}\rho^2 = 0, \tag{41}$$
$$S''' + 4\Gamma S' + 2\Gamma' S = 0. \tag{42}$$

The last equation together with (41) can be used to search for symmetries of specific reduced Ermakov systems with the traditional functional dependences ($\omega = \Omega(t)$ only). However, for *arbitrary* reduced Ermakov systems, that is, for completely arbitrary functions Γ, the only possibility is

$$S = 0. \tag{43}$$

This will be the choice if we are interested in symmetries valid for *all* non-reduced Ermakov systems with time-dependent frequencies, regardless its specific form. In this context, only quasi-invariance transformations are allowed.

The remaining condition (41), which can be integrated once yielding Pinney's [26] equation, is

$$\ddot{\rho} + \Omega^2 \rho = \frac{c}{\rho^3}, \tag{44}$$

where c is a constant. However, this is a non-linear equation, and a more fruitful approach for the study of symmetries is the linearizing transformation

$$a = \rho^2, \tag{45}$$

giving

$$\dddot{a} + 4\Omega^2 \dot{a} + 4\Omega\dot{\Omega} a = 0. \tag{46}$$

According to (26), the solution for this equation determines the symmetry generator

$$U = a\frac{\partial}{\partial t} + \frac{\dot{a}r}{2}\frac{\partial}{\partial r}. \tag{47}$$

Now, with the rescaling

$$\alpha = a/\psi^2, \qquad \tau = \int^t d\mu/\psi^2(\mu), \tag{48}$$

where ψ is any particular solution for the time-dependent harmonic oscillator equation

$$\ddot{\psi} + \Omega^2 \psi = 0, \tag{49}$$

we transform (46) into

$$\frac{d^3\alpha}{d\tau^3} = 0. \tag{50}$$

The general solution is (compare with (27))

$$\alpha = c_0 + c_1\tau + c_2\tau^2, \tag{51}$$

for constants c_0, c_1 and c_2. Taking separately each of these constants non-zero, we obtain three symmetry generators for arbitrary reduced Ermakov systems with frequency functions depending on time only. In the original, non-rescaled variables and using (47), the corresponding symmetry generators are

$$U_0 = \psi^2\frac{\partial}{\partial t} + \psi\dot{\psi}r\frac{\partial}{\partial r}, \tag{52}$$

$$\begin{aligned} U_1 = {} & \psi^2 \int^t \frac{d\mu}{\psi^2(\mu)}\frac{\partial}{\partial t} \\ & + \left(\frac{1}{2} + \psi\dot{\psi}\int^t \frac{d\mu}{\psi^2(\mu)}\right) r\frac{\partial}{\partial r}, \end{aligned} \tag{53}$$

$$\begin{aligned} U_2 = {} & \psi^2 \left(\int^t \frac{d\mu}{\psi^2(\mu)}\right)^2 \frac{\partial}{\partial t} \\ & + \left(1 + \psi\dot{\psi}\int^t \frac{d\mu}{\psi^2(\mu)}\right)\int^t \frac{d\nu}{\psi^2(\nu)} r\frac{\partial}{\partial r}. \end{aligned} \tag{54}$$

Calculating the Lie brackets, the result is

$$\begin{aligned} &[U_0, U_1] = U_0, \qquad [U_0, U_2] = 2U_1, \\ &[U_1, U_2] = U_2, \end{aligned} \tag{55}$$

which is the $sl(2, R)$ algebra. This shows that, for frequencies depending on time only, the symmetry group for arbitrary reduced Ermakov systems is $SL(2, R)$, the same as for arbitrary non-reduced Ermakov systems [17–21]. It is interesting to note that the algebra of the vector fields U_0, U_1 and U_2 is $sl(2, R)$ regardless the form of ψ (it does not need to be a solution of a time-dependent harmonic oscillator). In addition, we notice that U_0, U_1 and U_2 do not depend on the Ermakov invariant, being generators of *point* transformations also in the non-reduced space.

Following the same approach of Ref. [17], we can easily find more general classes of reduced Ermakov systems (with frequency functions not necessarily depending on time only) also admitting the $SL(2, R)$ group.

5. Application to a generalized Paul trap

Let us search for Lie point symmetries for the following class of Ermakov systems, written initially in Cartesian coordinates,

$$\ddot{x} + \left(\omega_0^2 - \frac{\vartheta(x\dot{y} - y\dot{x}, y/x)}{(x^2+y^2)^{3/2}}\right)x = \frac{L^2}{x^3}, \tag{56}$$

$$\ddot{y} + \left(\omega_0^2 - \frac{\vartheta(x\dot{y} - y\dot{x}, y/x)}{(x^2+y^2)^{3/2}}\right)y = 0, \tag{57}$$

where ω_0 and L are constants and ϑ is an initially arbitrary function of the indicated arguments. For constant ϑ, these are the equations for the secular motion for ions in the presence of a Paul trap [27] with equal secular frequencies. In this context, we call (56)–(57) the equations of motion for a generalized Paul trap. Paul traps are standard configurations used in ion trapping experiments [28]. They are given in terms of time-dependent external fields characterized by a fast and a slow time scales. After averaging over the fast time scale, we obtain autonomous equations describing the ion secular motion, as in (56)–(57). In this context, for constant ϑ, (56)–(57) describe the relative secular motion of two ions subject to their own electrostatic repulsion and to the trapping field. Also notice that, for $\omega_0 = 0$ and ϑ depending only on y/x, Eqs. (56)–(57) are a particular case of the Kepler–Ermakov systems, which are linearizable through point transformations [14]. Here we ask for the classes of functions ϑ for which the corresponding reduced Ermakov systems admit Lie point symmetries.

In polar coordinates, the generalized Paul trap can be cast in the Ermakov form (1)–(2) with

$$F(\theta) = \frac{L^2}{\cos^2\theta}, \qquad G(\theta) = -\frac{L^2 \sin\theta}{\cos^3\theta},$$
$$\omega^2 = \omega_0^2 - \frac{\vartheta(r^2\dot{\theta}, \tan\theta)}{r^3}. \tag{58}$$

Notice the generalized character of ω.

The associated Ermakov invariant is

$$I = \frac{1}{2}\left(r^2\dot{\theta}\right)^2 + \frac{L^2}{2\cos^2\theta}, \tag{59}$$

while the function H in (5) and the new angle φ in (6) are

$$H(\theta) = \frac{\sqrt{2I}}{\cos\theta}\left(1 - \frac{L^2}{2I} - \sin^2\theta\right)^{1/2}, \tag{60}$$

$$\varphi = \frac{1}{\sqrt{2I}} \arcsin\left(\frac{\sin\theta}{\sqrt{1 - L^2/2I}}\right). \tag{61}$$

Using the invariant, H and φ, we construct the reduced Ermakov system (7)–(8) with the function

$$A(r, \varphi, t) = -\omega_0^2 r + \frac{\sigma(\varphi)}{r^2} + \frac{2I}{r^3}, \tag{62}$$

where we have defined

$$\sigma(\varphi) = \vartheta\left(H(\theta), \tan\theta\right). \tag{63}$$

Let us search for symmetries of the type shown in Section 3. In practice, instead of looking for functions σ so that A in (62) is included in some of the subclasses 3.1, 3.2 or 3.3, a more convenient approach is to substitute A in the symmetry condition (25). This symmetry condition then gives

$$\left(\rho\dddot{\rho} + 3\dot{\rho}\ddot{\rho} + 4\omega_0^2\rho\dot{\rho}\right)r - \left(S\sigma' + \frac{3}{2}S'\sigma + \rho\dot{\rho}\sigma\right)r^{-2}$$
$$- \frac{1}{2}(S''' + 8IS')r^{-3} = 0. \tag{64}$$

We split the symmetry condition (64) into three equations, corresponding to different powers of r,

$$\rho\dddot{\rho} + 3\dot{\rho}\ddot{\rho} + 4\omega_0^2\rho\dot{\rho} = 0, \tag{65}$$
$$S\sigma' + \frac{3}{2}S'\sigma + \rho\dot{\rho}\sigma = 0, \tag{66}$$
$$S''' + 8IS' = 0. \tag{67}$$

For consistency, in Eq. (66) we must have

$$\rho\dot{\rho} = k, \tag{68}$$

for some constant k. Substituting this into (65), we get $k = 0$, so that

$$\rho = \rho_0, \tag{69}$$

for ρ_0 a constant.

Eq. (67) has the solution

$$S = S_0 + S_1\cos(2\sqrt{2I}\varphi) + S_2\sin(2\sqrt{2I}\varphi), \tag{70}$$

where S_0, S_1 and S_2 are constants, while Eq. (66) admits two classes of solution. For $S = 0$, σ is left arbitrary, reflecting the fact that the system is autonomous and hence invariant under time translations. For $S \neq 0$, (66) has the solution

$$\sigma = \frac{\sigma_0}{S^{3/2}}, \tag{71}$$

for some constant σ_0. In all cases, the symmetry generator is

$$U = \sqrt{2I}\Big(S_1\sin(2\sqrt{2I}\varphi)$$
$$- S_2\cos(2\sqrt{2I}\varphi)\Big)r\frac{\partial}{\partial r}$$
$$+ \left(S_0 + S_1\cos(2\sqrt{2I}\varphi) + S_2\sin(2\sqrt{2I}\varphi)\right)\frac{\partial}{\partial\varphi}$$
$$+ \rho_0^2\frac{\partial}{\partial t}. \tag{72}$$

Let us observe that we cannot take separately S_0, S_1 or S_2 non-zero, because these numerical constants participate into the equations of motion through the function

σ. In addition, according to the values of ρ_0, S_0, S_1 and S_2, the generator U falls into the classes 3.1, 3.2 or 3.3.

In the $S_1^2 + S_2^2 \neq 0$ case, the solution for S contains the Ermakov invariant, which is dependent on $r^2\dot{\theta}$. Excluding time translations, this implies σ and then ϑ depending on $r^2\dot{\theta}$. Hence, at least in the generalized Paul trap case, excluding mere time translations we have not found solutions for which σ is a function of θ only, a necessary condition to yield a Kepler–Ermakov system.

We can use the symmetry to further reduce the order of the system. In order to appreciate this in the case of the generalized Paul trap, we consider the case where $\rho_0 = 0$, leaving us with a generator of symmetries of the type shown in Section 3.3. This is certainly the less traditional situation, since for $\rho_0 = 0$ the symmetry transformation is definitely not a rescaling. In order to reduce the order of the dynamical system, we can introduce the differential invariants of $U^{[1]}$, the generator of the first extended group of symmetries. Obtaining $U^{[1]}$ and its differential invariants is routine calculation [16] and we omit the details. The result is that $I_1 = Sr^2$ and $I_2 = t$ as in Eq. (35) together with

$$I_3 = \frac{2S^2 r\dot{r}}{\dot{\varphi}} + SS'r^2, \qquad I_4 = \frac{S}{\dot{\varphi}} \tag{73}$$

are the differential invariants of $U^{[1]}$, for S given in Eq. (70). Notice that, along trajectories, $I_4 = I_1$, reflecting the fact that the reduced Ermakov system is of third order. Hence, only I_1, I_2 and I_3 are sufficient. Using Eqs. (35), (62), (70) and (71), we find that, on the solutions manifold,

$$\dot{I}_1 = I_3/I_1, \tag{74}$$

$$\dot{I}_2 = 1, \tag{75}$$

$$\dot{I}_3 = -2\omega_0^2 I_1^2 + 2\sigma_0\sqrt{I_1} + 4I\left(S_0^2 - S_1^2 - S_2^2\right) + \frac{3}{2}I_3^2/I_1^2. \tag{76}$$

Eq. (75) is just a triviality but (74) and (76) compose a second-order system, obtained via symmetry reduction. To solve this system, let us introduce the new variable

$$\tilde{r} = \sqrt{I_1}. \tag{77}$$

Using (74) and (76) we get

$$\ddot{\tilde{r}} = -\frac{dV}{d\tilde{r}} = -\omega_0^2\tilde{r} + \frac{\sigma_0}{\tilde{r}^2} + \frac{2I(S_0^2 - S_1^2 - S_2^2)}{\tilde{r}^3}, \tag{78}$$

for

$$V = V(\tilde{r}) = \frac{\omega_0\tilde{r}^2}{2} + \frac{\sigma_0}{\tilde{r}} + \frac{I(S_0^2 - S_1^2 - S_2^2)}{\tilde{r}^2}. \tag{79}$$

Thanks to the symmetry, we have obtained a one-dimensional autonomous potential system, clearly integrable by quadrature because of the existence of the energy integral

$$E = \frac{\dot{\tilde{r}}^2}{2} + V(\tilde{r}). \tag{80}$$

The effective potential V has one harmonic, one Kepler-like and one Ermakov-like term. If $\sigma_0 = 0$, a possibility not considered here because of its triviality, the harmonic term could be eliminated by a rescaling transformation [14,15]. In terms of the differential invariants, the energy integral is

$$E = \frac{I_3^2}{8I_1 I_4^2} + V(\sqrt{I_1}), \tag{81}$$

showing explicitly that the constant of motion is invariant under the symmetry group.

In general, a variety of behaviors can occur according to the signs of σ_0 and $S_0^2 - S_1^2 - S_2^2$. For instance, collapse is prevented provided $S_0^2 - S_1^2 - S_2^2 > 0$. Once $\tilde{r}(t)$ is obtained from the quadrature of the energy integral, the angle φ as a function of time follows from

$$\int_{\varphi_0}^{\varphi} \frac{d\phi}{S(\phi)} = \int_{t_0}^{t} \frac{d\tau}{\tilde{r}^2(\tau)}, \tag{82}$$

where $\varphi_0 = \varphi(t_0)$.

Some qualitative as well as quantitative information about the solutions can be found by considering the fixed points $\tilde{r} = \tilde{r}_0$ for which $dV/d\tilde{r} = 0$ at (78). These fixed points satisfy

$$\omega_0^2\tilde{r}_0^4 - \sigma_0\tilde{r}_0 - 2I\left(S_0^2 - S_1^2 - S_2^2\right) = 0. \tag{83}$$

The only non-trivial situation for which this quartic equation has simple closed form solutions is the case $S_0^2 - S_1^2 - S_2^2 = 0$. In this situation, there is the stationary solution

$$\tilde{r}_0 = \left(\frac{\sigma_0}{\omega_0^2}\right)^{1/3}. \tag{84}$$

If we take $\tilde{r} = \tilde{r}_0 + \tilde{r}_1 \exp(\lambda t)$ in the equation of motion and linearize with respect to $\tilde{r}_1$, we find the imaginary characteristic values $\lambda = \pm i\sqrt{3}\omega_0$, showing a linearly stable stationary solution.

Coming back to the original polar coordinates (r, θ), this shows that, for $S_0^2 - S_1^2 - S_2^2 = 0$, there are linearly stable orbits of the form

$$r = \frac{\tilde{r}_0}{\sqrt{S}}, \tag{85}$$

for $\tilde{r}_0 = (\sigma_0/\omega_0^2)^{1/3}$ and where S, expressed as a function of θ through (61) and (70), is given by

$$S = S_0 + S_1\left(1 - \frac{2\sin^2\theta}{1 - L^2/2I}\right) + \frac{2S_2\sin\theta}{\sqrt{1 - L^2/2I}}\left(1 - \frac{\sin^2\theta}{1 - L^2/2I}\right)^{1/2}. \tag{86}$$

The trajectories in terms of the time parameter associated to these orbits can be found from (82). For definiteness, let us consider the set of parameters $S_0 = S_1 > 0$, $S_2 = 0$, $\varphi_0 = 0$, yielding

$$r = \frac{\tilde{r}_0}{\sqrt{2S_0}}\sqrt{1 + \tilde{t}^2},$$
$$\theta = \arcsin\left(\frac{\sqrt{1 - L^2/2I}\,\tilde{t}}{\sqrt{1 + \tilde{t}^2}}\right), \tag{87}$$

where

$$\tilde{t} = 2S_0\sqrt{2I}\left(\frac{\omega_0^2}{\sigma_0}\right)^{2/3}(t - t_0). \tag{88}$$

This shows that, in the original, polar coordinates, the fixed point $\tilde{r} = \tilde{r}_0$ yields a trajectory asymptotically approaching a ballistic motion, $r \to \tilde{r}_0\tilde{t}/\sqrt{2S_0}$, $\theta \to \arcsin(\sqrt{1 - L^2/2I})$ as $\tilde{t} \to \infty$.

6. Conclusion

We presented a general treatment for Lie point symmetries of reduced Ermakov systems. We found three classes of reduced Ermakov systems possessing Lie point symmetries, all of them involving arbitrary functions. We have applied the results to a generalized Paul trap. From the theoretical viewpoint, the most important result we have found is the fact that the $SL(2, R)$ group is more exactly a property of the *reduced* Ermakov system with time-dependent frequencies, with the reduction process not perturbing the symmetry structure of the non-reduced system. Notice that this conclusion can be extended to a class of more general reduced Ermakov systems, with frequency functions depending also on dynamical variables, if we impose $SL(2, R)$ as a group of symmetries, as in Ref. [17]. For Lagrangian Ermakov systems, the existence of the Ermakov invariant follows from a dynamical symmetry. Then, at least for reduced Lagrangian Ermakov systems with traditional frequency functions, the symmetry structure can be split in two distinct parts: a dynamical symmetry leading to the Ermakov invariant, and the $SL(2, R)$ group for the reduced Ermakov system, the reduction being a consequence of the first symmetry. However, in the particular cases shown in Section 3, the reduced Ermakov systems can have symmetry generators containing a term certainly not associated to $SL(2, R)$, namely the $S(\varphi)$ term. This extra possibility is present in the generalized Paul trap of Section 5 and is used to integrate the equations of motion in a specific case.

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References

[1] V.P. Ermakov, Univ. Izv. Kiev 20 (1880) 1.
[2] J.R. Ray, J.L. Reid, Phys. Lett. A 71 (1979) 317.
[3] H.R. Lewis, Phys. Rev. Lett. 18 (1967) 510.
[4] F. Haas, Phys. Rev. A 65 (2002) 033603.
[5] J.E. Lidsey, Class. Quantum Grav. 21 (2004) 777.
[6] R.M. Hawkins, J.E. Lidsey, Phys. Rev. D 66 (2002) 023523.
[7] M.V. Ioffe, H.J. Korsch, Phys. Lett. A 311 (2003) 200.
[8] H.C. Rosu, Phys. Scr. 65 (2002) 296.
[9] V.P. Akulov, S. Catto, O. Cebioğlu, A. Pashnev, Phys. Lett. B 575 (2003) 137.
[10] A. Mostafazadeh, Phys. Lett. A 320 (2004) 375.
[11] M. Liang, Int. J. Theor. Phys. 42 (2003) 2881.
[12] F. Haas, J. Phys. A: Math. Gen. 35 (2002) 2995.
[13] J.L. Reid, J.R. Ray, J. Math. Phys. 21 (1980) 1583.
[14] F. Haas, J. Goedert, J. Phys. A: Math. Gen. 32 (1999) 2835.
[15] C. Athorne, C. Rogers, U. Ramgulam, A. Osbaldestin, Phys. Lett. A 143 (1990) 207.
[16] P.J. Olver, Applications of Lie Groups to Differential Equations, Springer, Berlin, 1986.

[17] J. Goedert, F. Haas, Phys. Lett. A 239 (1998) 348.
[18] P.G.L. Leach, Phys. Lett. A 158 (1991) 102.
[19] K.S. Govinder, C. Athorne, P.G.L. Leach, J. Phys. A: Math. Gen. 26 (1993) 4035.
[20] K.S. Govinder, P.G.L. Leach, J. Phys. A: Math. Gen. 27 (1994) 4153.
[21] K.S. Govinder, P.G.L. Leach, Phys. Lett. A 186 (1994) 391.
[22] F. Haas, J. Goedert, Phys. Lett. A 279 (2001) 181.
[23] A. Karasu (Kalkanli), H. Yildirim, J. Nonlinear Math. Phys. 9 (2002) 475.
[24] P.G.L. Leach, A. Karasu (Kalkanli), J. Nonlinear Math. Phys. 11 (2004) 269.
[25] C. Géronimi, Contribution a l'Integrabilité des Équations Différentielles Ordinaires Possédant des Symétries d'Invariance par Translation et de Redimensionnement, PhD. Thesis, Département Mathematiques, Université d'Orléans, France, 1998.
[26] E. Pinney, Proc. Amer. Math. Soc. 1 (1950) 681.
[27] W. Paul, H. Steinwedel, Z. Naturforsch. 8a (1953) 448.
[28] G. Baumman, T.F. Noemacher, Phys. Rev. A 46 (1992) 2682.