Espacios de Hilbert

Espacios de funciones con producto escalar

$(\,,\,) : X \times X \to$ Š

$(x,y) = \overline{(y,x)}$ $\quad (x,x) \geq 0$ i $(x,x)=0 \Rightarrow x=0$

$(\lambda x, y) = \lambda (x,y)$ $\quad (x, \lambda y) = \overline{\lambda}(x,y)$

$(x+y,z) = (x,z)+(y,z)$ $\quad (x,y+z) = (x,y)+(x,z)$

$\boldsymbol{C}_{[a,b]}$: espacio de las funciones complejas de variable real t tales que x:[a,b] → Š, continuas.

$: (x+y)(t) = x(t)+y(t)$

$: (x,y) = \int_a^b x(t)\overline{y(t)}dt$

$: \int_a^b x(t)dt = \int_a^b \mathrm{Re}\{x(t)\}dt + j\int_a^b \mathrm{Im}\{x(t)\}dt$

$\boldsymbol{L}_{(a,b)}$: funciones x:(a,b) → Š tales que $\int_a^b |x(t)|dt < \infty$

$\boldsymbol{L}^2_{(a,b)}$: funciones x:(a,b) → Š tales que $\int_a^b |x(t)|^2 dt < \infty$

$$||x|| = (x,x)^{1/2} = \sqrt{\int_a^b |x(t)|^2 dt}$$

$$d(x,y) = ||x-y|| = \sqrt{\int_a^b |x(t)-y(t)|^2 dt}$$

·desigualdad Cauchy-Swarz: $|(x,y)| \leq ||x|| \cdot ||y$

·desigualdad triangular: $||x+y|| \leq ||x|| + ||y$

Espacios de Hilbert

·Sucesión de Cauchy: $\forall \varepsilon > 0, \exists N | n,m > N \Rightarrow ||X_n - X_m|| < \varepsilon$

·Espacio completo sii toda sucesión de Cauchy es convergente.

·Espacio de Hilbert: espacio vectorial
con producto escalar
completo

·Š^n, ‘^{n}: son completos

$L^2_{(a,b)}$: completo (es la completación de $C_{[a,b]}$) con ∫ de Lebesgue

Familias ortonormales

$\{\varphi_i\}$tales que: $||\varphi_i|| = 1 \forall i$, $(\varphi_i, \varphi_j) = 0 \forall i \neq j$

método de Ortonormalización de Gram-Schmidt

teorema:

$\{x_k\}$ sucesión de vectores LI⇒ ∃ una sucesión ortonormal

$$\{x_k\} \to \overbrace{\{y_k\}}^{\text{ortogonal}} \to \overbrace{\{\phi_k\}}^{\text{ortonormal}}$$

$$y_n = x_n - \sum_{i=1}^{n-1} (x_n, \phi_i)\phi_i$$

$$\phi_n = \frac{y_n}{||y_n||} = \widehat{y_n}$$

Series de Fourier

$$x = \sum_{k=0}^{\infty} c_k \Phi_k$$

$\{\Phi_k\}$sucesión ortonormal $\quad c_k = (x, \Phi_k) = \overline{(\Phi_k, x)}$

Serie de Fourier de X respecto de $\{\Phi_k\}$ con coef. de Fourier c_k

$$x = \sum_{k=0}^{\infty} c_k \Phi_k \text{ si } ||\Phi_i|| = 1 \forall i$$

$$\text{si no:} x = \sum_{k=0}^{\infty} \frac{(x,\psi_k)}{||\psi_k||}\frac{\psi_k}{||\psi_k||} = \sum_{k=0}^{\infty} \frac{(x,\psi_k)}{||\psi_k||^2}\psi_k$$

Desigualdad de Bessel:

$$\sum_{k=0}^{\infty} |(x,\phi_k)|^2 \leq ||x||^2 \to \lim_{k\to\infty} c_k = 0$$

Teorema de Parseval:

$\{\phi_k\}$ortonormal, es completa sii:

$$\forall x \in X, \sum_{k=0}^{\infty} |(x,\phi_k)|^2 = ||x||^2$$

Aproximación mediante combinaciones lineales

$$||x - \sum_{k=0}^{n} d_k \Phi_k|| \geq ||x - \sum_{k=0}^{n} c_k \Phi_k||$$

Series de Fourier trigonométricas

$$x \in L^2_{(-\pi,\pi)} : x = \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos kt + b_k \sin kt)$$

$$a_k = \int_{-\pi}^{\pi} x(t)\cos kt dt \qquad b_k = \int_{-\pi}^{\pi} x(t)\sin kt dt$$

Convergencia

La serie de fourier de x converge a $\frac{x(t^+)+x(t^-)}{2}$

y si ∃x'(t) ⇒ converge a x(t)

Convergencia uniforme

$x : [-\pi,\pi], x(-\pi) = x(\pi), continua : \Rightarrow$

su serie de Fourier converge **uniformemente**.

Derivación

$$\frac{d}{dt}\left(\frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos kt + b_k \sin kt)\right) = \sum_{k=1}^{\infty} k(b_k \cos kt - a_k \sin kt)$$

Integración

$$\int_{-\pi}^{t}\left(\frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos k\tau + b_k \sin k\tau)\right)d\tau =$$

$$\frac{a_0}{2}(t+\pi) + \sum_{k=1}^{\infty}\frac{1}{k}(a_k \sin kt - b_k(\cos kt - \cos k\pi))$$

Órdenes de magnitud

si x : ‘ → ‘ periódica continua con derivadas: x',x'',..,$x^{p-1)}$, $p \geq$

y tal que $x^{p)} \in L^2_{(-\pi,\pi)} \Rightarrow |a_k|, |b_k| \leq \frac{\varepsilon_k}{k^p}$

$k = 1,2,3...$ donde $\varepsilon_k \to 0$ cuando k → ∞

La transformada continua de Fourier

$X(f)=\int_{-\infty}^{\infty}x(t)e^{-j2\pi ft}dt$ $\quad x(t)\in L^2_{(-\infty,\infty)}$ o $x(t)\in L_{(-\infty,\infty)}$

$x(t)=\int_{-\infty}^{\infty}X(f)e^{j2\pi ft}df$ $\quad$ en intervalo finito: finitos max y mins, y saltos.

T.d.F. para señales periódicas

$$x(t)=\sum_{k=-\infty}^{+\infty}a_k e^{j2\pi\frac{k}{T_0}t}\longleftrightarrow X(f)=\sum_{k=-\infty}^{+\infty}a_k\delta(f-\frac{k}{T_0})$$

a_k : coeficiente de Fourier $=\frac{1}{T_0}X(\frac{k}{T_0})$

Propiedades

$$\alpha x+\beta y\longleftrightarrow\alpha X+\beta Y$$
$$x^*(t)\longleftrightarrow X^*(-f)$$
$$x(t-t_0)\longleftrightarrow X(f)e^{-j2\pi ft_0}$$
$$X(f-f_0)\longleftrightarrow F\{x(t)e^{j2\pi f_0 t}\}$$
$$x(-t)\longleftrightarrow X(-f)$$
$$x(at)\longleftrightarrow\frac{1}{|a|}X(af)$$
$$x(t)*y(t)\longleftrightarrow X(f)Y(f)$$
$$x(t)y(t)\longleftrightarrow X(f)*Y(f)$$
$$x'(t)\longleftrightarrow j2\pi f\cdot X(f)$$
$$X'(f)\longleftrightarrow -j2\pi t\cdot x(t)$$
$$\int_0^t x(\tau)d\tau\longleftrightarrow\frac{1}{2}X(0)\delta(f)+\frac{X(f)}{j2\pi f}$$
$$X(t)\longleftrightarrow x(-f)$$
$$\int_{-\infty}^{\infty}|x(t)|^2dt=\int_{-\infty}^{\infty}|X(f)|^2dt$$

Simetrías:

$$x(t)=\underbrace{x_{PAR}(t)}+\underbrace{x_{IMPAR}(t)}\longleftrightarrow\underbrace{\mathrm{Re}\{X(f)\}}+\underbrace{j\,\mathrm{Im}\{X(f)\}}=X(f)$$

$x_{PAR}(t)$	$x_{IMPAR}(t)$	$\mathrm{Re}\{X(f)\}$	$j\,\mathrm{Im}\{X(f)\}$
$\updownarrow F$	$\updownarrow F$	$\updownarrow F^{-1}$	$\updownarrow F^{-1}$
$X(f)$	$X(f)$	$x(t)$	$x(t)$
real	im.pura	par	impar
y par	e impar	real	real

Transformadas básicas

$$\delta(t)\longleftrightarrow 1$$
$$\delta(t-t_0)\longleftrightarrow e^{-j2\pi t_0 f}$$
$$1\longleftrightarrow\delta(f)$$
$$\cos 2\pi f_0 t\longleftrightarrow\frac{1}{2}(\delta(f-f_0)+\delta(f+f_0))$$
$$\sin 2\pi f_0 t\longleftrightarrow\frac{1}{2j}(\delta(f-f_0)-\delta(f+f_0))$$
$$\frac{1}{a}\Pi\left(\frac{t}{a}\right)\longleftrightarrow T\mathrm{sinc}(Tf)$$
$$u(t)e^{at}\longleftrightarrow\frac{1}{j2\pi f-a}$$
$$\frac{t^{n-1}}{(n-1)!}u(t)e^{at}\longleftrightarrow(\frac{1}{j2\pi f-a})^n$$
$$sgn(t)\longleftrightarrow\frac{1}{j\pi f}$$
$$u(t)\longleftrightarrow\frac{1}{2}\delta(f)+\frac{1}{j2\pi f}$$
$$\sum\delta(t-nT)\longleftrightarrow\frac{1}{T}\Sigma\delta(f-\frac{n}{T})$$
$$\triangle(\frac{t}{T})\longleftrightarrow T\sin c^2(Tf)$$
$$x(t)\cos(2\pi f_0 t)\longleftrightarrow\frac{X(f-f_0)}{2}+\frac{X(f+f_0)}{2}$$
$$e^{j2\pi f_0 t}\longleftrightarrow\delta(f-f_0)$$

Ventanas

$$X(f)=F\{x(t)\cdot\Pi(\tfrac{t}{T})\}=X(f)*T\sin c(Tf)$$

Acotaciones

$L_{(-\infty,\infty)}:|X(f)|\le\int_{-\infty}^{+\infty}|x(t)|dt<\infty$

$L^2_{(-\infty,\infty)}:$

$:|\int ur|^2\le\int|u|^2\int|r|^2$

$:|\int x(t)dt|\le\int|x(t)|dt$

Desarrollos en serie de Fourier

$$x(t)=\sum_{k=-\infty}^{+\infty}x_b(t-kt_0)=x_b(t)*\sum_{k=-\infty}^{+\infty}\delta(t-kt_0)$$

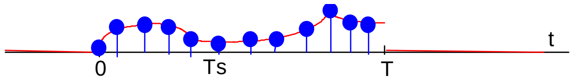
$$F\{x(t)\}=X(f)=X_b(f)\frac{1}{T_0}\sum_{k=-\infty}^{+\infty}\delta(f-\frac{k}{T_0})=\sum_{k=-\infty}^{+\infty}\underbrace{\frac{X_b(\frac{n}{T_0})}{T_0}}_{c_k}\delta(f-\frac{k}{T_0})$$

$$X(f)=\sum_{k=-\infty}^{+\infty}c_k\delta(f-\frac{k}{T_0})\longleftrightarrow x(t)=\sum_{k=-\infty}^{+\infty}c_k e^{j\frac{2\pi}{T_0}kt}$$

si x(t) es real:

:

I. La serie discreta de Fourier



Queremos aproximar su transformada de Fourier.

$$x[k] = x[k+N], k \in Z$$

$$T_s = \frac{T}{N}$$

$$\{\phi_n = e^{j2\pi \frac{n}{T} t}\} \text{ es base ortogonal } L^2_{(0,T)}$$

$$t_k = k\frac{T}{N} \qquad X[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}kn}$$

$$\omega = e^{j\frac{2\pi}{N}} \qquad x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k] e^{j\frac{2\pi}{N}kn}$$

$$X[k] = \sum_{n=0}^{N-1} x[n]\cdot\omega^{-nk}$$

$$x[n] = \sum_{k=0}^{N-1} X[k]\cdot\omega^{nk}$$

<u>Propiedades</u>

$$\alpha x[k] + \beta y[k] \leftrightarrow \alpha X[n] + \beta \;\; [n$$

$$x[\;\;] \in \quad \Rightarrow X\; N - \;\;] = \;\; ^*[\;\;]$$

$$X\; k] \qquad Nx\;(- \;\;) \qquad\qquad] : \text{dualidad}$$

$$[(\;\; - m \;\; \bmod N \;\; \leftrightarrow X\; n] \quad ^{-n}$$

$$x[\;\;]\omega\;^{mk} \qquad X[\; n - \;\;\;) \qquad\qquad]$$

$$\sum_{=0}^{-1} \;[(\;\; - i \;\; \bmod N \;\; y[\;] \leftrightarrow \;\; [n\;\; Y[\;\;]$$

$$x\;\; k]\;\; [k \;\; \leftrightarrow \sum_{=0}^{-1} \;\; [(\;\; - i \;\; \bmod N \;\; Y[\;]$$

$$\sum_{=0}^{-1} x[\;\;]| \;\; = \;\; \sum_{=0}^{-1} X[\;\;]|$$

La transformada de fourier discreto

$$X(f) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j2\pi fn}$$

$$x[n] = \int_0^1 X(f) e^{j2\pi fn} df$$

(Algunas cosas que hay que saber)

$$\sum_{k=m}^{n} \frac{1}{2^k} = 2\cdot 2^{-m} - 2^{-n}$$

$$\sum_{k=m}^{n} r^k = \frac{r^{f+1} - r^i}{r-1}$$

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt$$

$$\Gamma(k | k \in \text{Œ}) = (k-1)!$$

$$\cos\beta = \frac{1}{2}(e^{j\beta} + e^{-j\beta})$$

$$\sin\beta = \frac{1}{2}(e^{j\beta} - e^{-j\beta})$$

$$\cos(x) = \sin(x + \pi/2)$$

$$\cos(x+\alpha) = \cos\alpha\cos x - \sin\alpha\sin x$$

$$\sin 2x = 2\sin x\cos x$$

$$\cos 2x = \cos^2 x - \sin^2 x$$

$$\sin(a+b) = \cos a\sin b + \sin a\cos b$$

$$\int x\cos kx\,dx = \frac{x\sin kx}{k} + \frac{\cos kx}{k^2}$$

$$\int x\sin kx\,dx = \frac{-x\cos kx}{k} + \frac{\sin kx}{k^2}$$

I. Funciones de variable compleja

$f \;:\; D \subset$ Š $\to$ Š

$z \mapsto s(z) = u(z) + jv(z) \qquad z = x + jy$

$|z| = \sqrt{x^2 + y^2} = \sqrt{z\,\bar{z}}$

D : dominio donde está definida la función

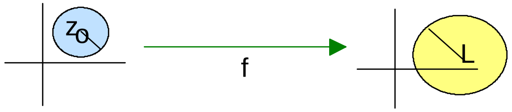
: es abierto y conexo.

Límites

$f : D \subset$ Š $\to$ Š

: tiene límite L en $z_o \in D$ si $\forall \varepsilon > 0, \forall \delta > 0$

: $|z - z_0| < \delta \Rightarrow |f(z) - l| < \varepsilon$



$$\lim_{z \to z_0} f(z) = L \Leftrightarrow \begin{array}{l} \lim_{z\to z_0} \operatorname{Re}\{f(z)\} = \operatorname{Re}\{L\} \\ \lim_{z\to z_0} \operatorname{Im}\{f(z)\} = \operatorname{Im}\{L\} \end{array}$$

$f : D \subset$ Š $\to$ Š es continua en $z_0 \in D$ si

: $\lim_{z \to z_0} f(z) = f(z_0)$

Derivación

$f : D \subset$ Š $\to$ Š es derivable en $z_0 \in D$ si $\exists$ el límite

: $\lim_{z \to z_0} \dfrac{f(z) - f(z_0)}{z - z_0} = f'(z_0)$

f y g son derivables en z_0

: $\cdot(\lambda f + \mu g)'(z_0) = \lambda f'(z_0) + \mu g'(z_0)$

: $\cdot(f \cdot g)'(z_0) = f'(z_0)g(z_0) - f(z_0)g'(z_0)$

: $\cdot(\frac{1}{g})'(z_0) = \dfrac{-g'(z_0)}{g^2(z_0)}$ si $g(z) \neq 0 \in B_R(z_0)$

observación:

$$\lim_{h\to 0} \frac{f(z_0 + h) - f(z_0)}{h} =$$

$$\lim_{h\to 0} \frac{u(x_0 + h, y_0) + jv(x_0 + h, y_0) - u(x_0, y_0) - v(x_0, y_0)}{h} =$$

$$= \frac{\partial u}{\partial x}(x_0, y_0) + j\frac{\partial v}{\partial x}(x_0 y_0)$$

$$\lim_{h\to 0} \frac{f(z_0 + jh) - f(z_0)}{jh} =$$

$$\lim_{h\to 0} \frac{u(x_0, y_0 + h) + jv(x_0, y_0 + h) - u(x_0, y_0) - v(x_0, y_0)}{jh} =$$

$$= \frac{\partial u}{\partial y}(x_0, y_0) - j\frac{\partial v}{\partial y}(x_0 y_0)$$

Condiciones de Cauchy-Riemann

Si f es derivable en Zo, entonces satisface: $u_x = v_y$, $u_y = -v_x$

entonces: $f'(z) = \dfrac{\partial}{dx}u(x, y) + j\dfrac{\partial}{\partial x}v(x, y)$

Funciones harmónicas y ecuación de LaPlace

$\Delta\mu = \dfrac{\partial^2 \mu}{\partial x^2} + \dfrac{\partial^2 \mu}{\partial y^2}$ si $\Delta\mu = 0 \Rightarrow \mu$ es armónica

$f = u + jv$ derivable$\Rightarrow u, v$ son armónicas conjugadas.

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\frac{\partial u}{\partial x} \overset{CR}{=} \frac{\partial}{\partial x}\frac{\partial v}{\partial y} \overset{TS}{=} \frac{\partial}{\partial y}\frac{\partial v}{\partial x} \overset{CR}{=} \frac{\partial}{\partial y}(-\frac{\partial u}{\partial y}) = -\frac{\partial^2 x}{\partial y^2}$$

CR : condiciones de Cauchy-Riemann

TR : Teorema de Swarz (cruzadas)

Trayectorias ortogonales

:

Funciones elementales

Polinomios:

$p(z) = a_0 + a_1 z + \cdots + a_n z^n$

$z^n = (x + jy)^n = r^n e^{jn\theta} = r^n(\cos n\theta + j\sin n\theta)$

Exponencial: $e^z = e^{x+jy} = e^z(\cos y + j\sin y)$

$(e^z)' = e^z$

Trigonométricas:

$\cos z = \frac{1}{2}(e^{jz} + e^{-jz})$

$\sin z = \frac{1}{2}(e^{jz} - e^{-jz})$

$\tan z = \dfrac{\sin z}{\cos z}$

Hiperbólicas:

$\cosh z = \frac{1}{2}(e^z + e^{-z})$

$\sinh z = \frac{1}{2}(e^z - e^{-z})$

$\tanh z = \dfrac{\sinh z}{\cosh z}$

Logaritmo: $\ln(z) = \ln(re^{j\theta}) = \ln r + j(\theta + k2\pi), k \in Z$

Derivación en polares

$$f(z)=f(re^{j\theta})=u(r,\theta)+jv(r,\theta)$$

$$f'(z)=\frac{1}{e^{j\theta}}[u_r+jv_r]=\frac{1}{re^{j\theta}}[v_\theta-ju_\theta]$$

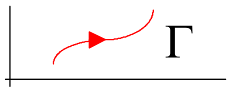
$$:\ u_r=\frac{1}{r}v_\theta\ :\ v_\theta=-\frac{1}{r}u_\theta$$

Función potencial

$$f(z)=z^c=e^{c\ln z}=e^{c\ln r}e^{cj(\theta+2\pi k)}$$

Integración

$$\int_\Gamma f(z)dz$$



$$\int_\Gamma f(z)dz=\int_\Gamma[u_{(x,y)}+jv_{(x,y)}][dx+jdy]=$$

$$=\int_\Gamma <\overbrace{(u,-v)}^{F_R},(dx,dy)>+j\int_\Gamma <\overbrace{(v,u)}^{F_I},(dx,dy)>=\int_\Gamma \vec{F_R}\cdot\vec{dl}+j\int_\Gamma \vec{F_I}\cdot\vec{dl}$$

$$\int_\Gamma f(z)dz=\int_a^b f(z(t))\cdot z'(t)dt$$

Propiedades

$$\int_\Gamma(f+g)=\int_\Gamma f+\int_\Gamma g$$

$$\int_\Gamma f=\int_{\Gamma 1} f+\int_{\Gamma 2} f$$

$$\int_\Gamma f=\int_{-\Gamma} f$$

$$\left|\int_\Gamma f(z)dz\right|\le ML$$

$$L=long\Gamma$$

$$|f(z)|\le M,\forall z\ \varepsilon\ \Gamma$$

Teorema de Cauchy

sea f analítica en un dominio D (simplemente conexo y abierto)

$$\oint_\Gamma f(z)dz=0$$

Si C es un contorno simple cerrado que no pasa por el punto a y n es un numero entero se verifica:

$$\int_C(z-a)^n dz=\begin{cases}0: & \text{si } n\ne -1\\ 0: & \text{si } n=-1 \text{ y } a \text{ está fuera de C}\\ 2\pi j: & \text{si } n=-1 \text{ y } a \text{ está dentro de C}\end{cases}$$

la de integración de Cauchy

$$f^{(n)}(z_o)=\frac{n!}{2\pi j}\int_\Gamma\frac{f(z)}{(z-z_o)^{n+1}}dz \quad\Rightarrow f(z_o)=\frac{1}{2\pi j}\int_\Gamma\frac{f(z)}{(z-z_o)}dz$$

Regla de Barrow

$$\int_a^b f(z)dz=F(b)-F(a)$$

$$\frac{d}{dz}F=f(z)$$

Series

Taylor: $f(z)=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}z^n$

$$e^z=\sum_{n=0}^{\infty}\frac{z^n}{n!}$$

Ejemplos: $\cos x=1-\frac{x^2}{2}+..$

$$\sin z=x-\frac{x^3}{3!}+..$$

$$f(z)=\sum_{n=0}^{\infty}c_n(z-z_o)^n$$

$$c_n=\frac{f^{(n)}_{(z_0)}}{n!}=\frac{1}{2\pi j}\int_\Gamma\frac{f(z)}{(z-z_o)^{n+1}}dz$$