Processat del senyal

Temari

Professorat i avaluació

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2n control:	19/22 Maig
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Nota:	max{E.F, 0.6E.F.+0.2P1+0.2P2}

Introducció

Objectius

"Estudi de les tècniques i algorismes de processat digital del senyal, enfocat a aplicacions en comunicacions i tractament d'imatge i veu"

Aplicacions

- Anàlisi espectral
- Caracterització paramètrica del senyal
- Filtrat adaptatiu
- Codificació i compressió de la imatge

Notació matricial

$x(n)$: seqüència

$$\underline{x} : \text{vector} = \begin{vmatrix} x(0) \\ \vdots \\ x(N-1) \end{vmatrix}$$

$$\underline{x}^T = \begin{vmatrix} x(0) & \cdots & x(N-1) \end{vmatrix}$$

$\underline{\underline{x}}$: matriu

$$\underline{h}^H \underline{x} = \sum_{n=0}^{N-1} h^*(n)x(n)$$

$$\|\underline{x}\|^2 = \sum_{n=0}^{N-1} |x(n)|^2 = \underline{x}^H \underline{x}$$

desigualtat de Schwartz

$$|\underline{h}^H \underline{x}|^2 \leq \|\underline{h}\|^2 \|\underline{x}\|^2 \qquad \text{sii } h(n) = ax(n) \Rightarrow \textit{igualtat}$$

Coherència

$$|g|^2 = \frac{|\underline{h}^H \underline{x}|^2}{\|\underline{h}\|^2 \|\underline{x}\|^2} \leq 1 \qquad \text{sii } h(n) = ax(n) \Rightarrow g = 1$$

Run-time vector

$$\underline{X}_n^T = [x(n), x(n-1), \ldots, x(n-Q+1)]$$

Processat del senyal

Autocorrelació

Distància entre senyals

$$D_{xy}=\sum_{n=0}^{N-1}|x(n)-y(n)|^2=(\underline{x}-\underline{y})^H(\underline{x}-\underline{y})=\|\underline{x}\|^2+\|\underline{y}\|^2-\underline{x}^H\underline{y}-\underline{y}^H\underline{x}$$

Autocorrelació de x(n)

$$r(m)=\frac{1}{M}\sum_n x^*(n-m)x(n)=\frac{1}{M}\sum_n x^*(n)x(n+m)$$

Matriu de correlació a partir de $\underline{X}_n$

$$\underline{R}=\frac{1}{M}\sum_n \underline{X}_n\underline{X}_n^*\cong\begin{bmatrix} r(0) & r(1) & \cdots & r(Q-1)\\ r(-1) & r(0) & \cdots & r(Q-2)\\ \vdots & \vdots & \ddots & \vdots\\ r(-Q+1) & r(-Q+2) & \cdots & r(0)\end{bmatrix}$$

Filtrat

$$y(n)=\sum_{q=0}^{Q-1}h^*(q)x(n-q)=\underline{h}^H\cdot\underline{x}_n=\begin{vmatrix}h(0)^* & \cdots & h^*(Q-1)\end{vmatrix}\begin{vmatrix}x(0)\\ \vdots\\ x(Q-1)\end{vmatrix}$$

$$|y(n)|^2=\underline{h}^H\underline{X}_n\underline{X}_n^H\underline{h}$$

$$r_y(0)=\frac{1}{M}\sum_n|y(n)|^2=\underline{h}^H\underline{R}\underline{h}$$

$$r_y(0)\le 0$$

Transitori

$x(n)$, N mostres $\quad h(n)$, Q mostres (filtre FIR)

$$\underline{y}_T(n)=\Big[\underbrace{y(0),y(1),\ldots,y(Q-1)}_{\text{pre-transitori}\ _{\text{Q mostres}}},\underbrace{y(Q),,\ldots,y(N-1)}_{\text{règim transitori}\ _{\text{N-Q mostres}}},\underbrace{y(N),\ldots,y(N+Q-1)}_{\text{post-transitori}\ _{\text{Q mostres}}}\Big]$$

$$E_y=\underline{y}^H\underline{y}=\underline{y}^T\underline{y}^*\qquad\qquad P_y=\frac{1}{M}\underline{y}^H\underline{y}=\underline{h}^H\frac{1}{M}\underline{X}\,\underline{X}^H\underline{h}$$

$$\underline{X}=\begin{bmatrix} x(0) & x(1) & \cdots & x(Q-1) & \cdots & x(N-1) & 0 & \cdots & 0\\ 0 & x(0) & \cdots & x(Q-2) & \cdots & x(N-2) & x(N-1) & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & 0\\ 0 & 0 & \cdots & x(0) & \cdots & x(N-Q) & x(N-Q+1) & \cdots & x(N-1)\end{bmatrix}$$

$$\underline{y}=\underline{h}^H\underline{X}$$

Sinus en soroll i el lema de la inversa

$$s(n)=A_0e^{j(2pf_0n+q)}=Ae^{j2pf_0n}$$

$$\underline{S}_n=Ae^{j2pf_0n}\begin{bmatrix}1\\ e^{-j2pf_0}\\ \vdots\\ e^{-j2pf_0(Q-1)}\end{bmatrix}=Ae^{j2pf_0n}\underline{S}_0$$

$$\underline{R}_0=\frac{1}{M}\sum_n\underline{S}_n\underline{S}_n^H=|A|^2\underline{S}_o\underline{S}_o^H$$

$$y(n)=s(n)+w(n)$$

$$\underline{R}=|A|^2\underline{S}_0\underline{S}_0^H+s^2\underline{I}$$

Si existeixen M<Q sinus ortogonals, existiran M termes iguals.

$$\underline{R}=\sum_{m)=1}^{M}|A_m|^2\underline{S}_m\underline{S}_m^H+s^2\underline{I}=\underline{R}_S+s^2\underline{I}$$

Lema de la inversa

si $\underline{R}=\underline{G}\underline{G}^H+\underline{R}_a\Rightarrow$

$$\underline{R}^{-1}=\underline{R}_a^{-1}-\underline{R}_a^{-1}\underline{G}(\underline{I}+\underline{G}^H\underline{R}_a^{-1}\underline{G})^{-1}\underline{G}^H\underline{R}_a^{-1}$$

Minimització amb restriccions

Disseny de filtres o sistemes pot formular-se sempre com a minimització d'una funció objectius, subjecta a restriccions. Per processos gaussians i sistemes lineals, aquesta funció és l'error quadràtic mig (MSE).

$$MSE = x = \underline{a}^H \underline{C}\underline{a}$$

restriccions: $\Phi \underline{a} = \underline{f}$

$$MSE_{min} = \underline{f}^H (\Phi \underline{C}^{-1} \Phi^H)^{-1} \underline{f} \text{ amb } \underline{a} = \underline{C}^{-1} \Phi^H (\Phi \underline{C}^{-1} \Phi^H)^{-1} \underline{f}$$

Autovalors i autovectors

$\underline{R}\underline{e}_q = l_q \underline{e}_q$

$\underline{e}_q$: autovector

l_q : autovalor

Per ser $\underline{R}$ hermítica (dim QxQ) existeixen Q autovectors diferents que permeten fer la descomposició següent:

$$\underline{R} = \sum_{q=1}^{Q} l_q \underline{e}_q \underline{e}_q^H = [\underline{e}_1 \ \cdots \ \underline{e}_Q] \begin{bmatrix} l_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & l_q \end{bmatrix} \begin{bmatrix} \underline{e}_1^H \\ \vdots \\ \underline{e}_Q^H \end{bmatrix} = \underline{E}\underline{D}\underline{E}^H$$

$$\underline{I} = \underline{E}^H \underline{E} = \underline{E}\underline{E}^H$$

Propietats

- Si la matriu és definida positiva, tots els seus autovalors ho són.
- Si la matriu R és de rang M<Q, el número d'autovalors zero és Q-M
- El determinant de la matriu R és el producte dels seus autovalors, i la seva traça és la suma dels autovalors.
- $\ln\{\underline{R}\} = \underline{E} \ln\{\underline{D}\} \underline{E}^H$
- $e^{\{\underline{R}\}} = \underline{E} e^{\{\underline{D}\}} \underline{E}^H$
- $\underline{R}^m = \underline{E}\underline{D}^m \underline{E}^H$
- $\underline{R}^{-1} = \underline{E}\underline{D}^{-1} \underline{E}^H$

Valors singulars

$\dim\{\underline{R}\} = QxQ$

$\dim\{\underline{X}\} = QxN$

$\underline{R} = \underline{X}\,\underline{X}^H$

$$\underline{X} = \sum_{q=1}^{Q} f_q \underline{u}_q \underline{v}_q^H = [\underline{u}_1 \ \cdots \ \underline{u}_Q] \begin{bmatrix} f_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & f_q \\ 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} \underline{v}_1^H \\ \vdots \\ \underline{v}_N^H \end{bmatrix} = \underline{U}\Phi\underline{V}^H$$

$$\underline{I} = \underline{Q}^H \underline{Q} = \underline{V}^H \underline{V}$$

$$(\underline{X}\,\underline{X}^H)\underline{u}_q = f_q^2 \underline{u}_q, q = 1..Q \qquad (\underline{X}^H \underline{X})\underline{v}_q = \begin{cases} f_q^2 \underline{v}_q & q = 1..Q \\ 0 & q = Q+1...N \end{cases}$$

$$\underline{R} = \underline{U}\Phi^2 \underline{U}^H = \underline{U}\underline{D}\underline{U}^H$$

Sistemas adaptativos

Introducción

Los diseños óptimos dan lugar a degradación de las prestaciones cuando las condiciones del escenario cambian.

De este modo, cuando un sistema de tratamiento de señal es capaz de adaptarse automáticamente al entorno de señal, se dice que es **adaptativo**.

$$\text{Amplificador}: d(n) = g(n)x(n)$$

En el sistema de apredizaje, se dispone de la señal de referencia r(n), o simplemente de su potencia Pr, basta pues calcular la potencia de la señal de salida según:

$$P_d(n) = bP_d(n-1) + (1-b)d^2(n)$$
$$e(n) = P_r - P_d(n)$$
$$g(n+1) = g(n) + m e(n)$$

MSE y métodos del gradiente

$d(n)$: señal de referecia

$\underline{a}_n$: respuesta impulsional el filtro FIR

$\underline{X}_n$: vector de datos

$$MSE = x(\underline{a}_n) = E\left\{\left|d(n) - \underline{a}_n^H \underline{X}_n\right|^2\right\}$$

$$x(\underline{a}_n) = P_d + \underline{a}_n^H \underline{\underline{R}}\underline{a}_n - \underline{a}_n^H \underline{P} - \underline{P}^H \underline{a}_n$$
$$\frac{\partial x(\underline{a}_n)}{\partial \underline{a}_n^H} = \nabla x = \underline{\underline{R}}\underline{a}_{opt} - \underline{P} = 0 \Rightarrow \underline{a}_{opt} = \underline{\underline{R}}^{-1}\underline{P}$$
$$x(\underline{a}_n) = x_{min} + (\underline{a}_n - \underline{a}_{opt})^H \underline{\underline{R}}(\underline{a}_n - \underline{a}_{opt})$$
$$x_{min} = P_d - \underline{P}^H \underline{\underline{R}}\underline{P}$$

$$\underline{a}_{n+1} = \underline{a}_n - m\nabla x(\underline{a}_n) \qquad x(\underline{\tilde{a}}_n) = x_{min} + \underline{\tilde{a}}_n^H \underline{\underline{R}}\underline{\tilde{a}}_n$$

$$\nabla x(\underline{\tilde{a}}_n) = \underline{\underline{R}}\underline{\tilde{a}}_n$$

$$x = x_{min} + \underline{\tilde{a}}_n^H (\underline{\underline{E}}\underline{\underline{\Lambda}}\underline{\underline{E}}^H)\underline{\tilde{a}}_n = x_{min} + \underline{z}_n^H \underline{\underline{\Lambda}}\underline{z}_n^H = x_{min} + \sum_{i=1}^{Q} l_i\,|z_n(i)|^2$$

$$\underline{z}_n = \underline{E}^H(\underline{a}_n - \underline{a}_{opt})$$

Diseño y convergencia de algoritmos del gradiente

$$\underline{a}_{n+1} = \underline{a}_n - m\nabla x_n = \underline{a}_n - m(\underline{\underline{R}}\underline{a}_n - \underline{P})$$

$$m < \frac{2}{Traza\{\underline{\underline{R}}\}} < \frac{2}{l_{max}} \qquad P_x(n+1) = bP_x(n) + (1-b)\underline{X}_n^H \underline{X}_n$$

$$m = \frac{2a}{P_x(n)} \qquad \frac{1}{1-b}: \text{número de muestras de la memoria}$$

Usualmente se establece un umbral mínimo P_{xo} .

$$N_c = \frac{2{,}30}{-\ln(1-m\,l_{min})} = \frac{2{,}30}{-\ln\left(1-2a\,\frac{l_{min}}{l_{max}}\right)} \approx \frac{2{,}30}{2a}\frac{l_{max}}{l_{min}}$$

$$TN_c = \frac{2{,}30}{2a}Q\frac{P_s}{P_w} = \frac{2{,}30}{2a}QSNR$$

El algoritmo LMS (Least Mean Square algorithm)

$$y(n) = \underline{a}^H \underline{X}_n$$
$$e(n) = d^*(n) - y^*(n) \qquad m < \frac{2}{\underline{X}_n^H \underline{X}_n}$$
$$\underline{a}_{n+1} = \underline{a}_n + m\underline{X}_n e^*(n)$$

$$E\{\underline{a}_n\} = \underline{a}_{opt}$$
$$\underline{\tilde{a}}_n = \underline{a}_n - \underline{a}_{opt} : \text{error de coeficientes}$$
$$\underline{\tilde{a}}_{n+1} = \underline{\tilde{a}}_n - m e^*(n)\underline{X}_n$$

$$d(n) = w(n) + \underline{a}_{opt}^H \underline{X}_n$$
$$e(n) = w(n) - \underline{\tilde{a}}_{opt}^H \underline{X}_n$$

$\underline{\underline{\Sigma}}_a = E\{\underline{\tilde{a}}_{opt}\underline{\tilde{a}}_{opt}^H\} \cong \frac{m}{2} x_{min} \underline{\underline{I}}$: covarianza delos coeficientes

$x(\text{n}) = x_{\min} + \underline{\tilde{a}}_n^H \underline{\underline{R}}\underline{\tilde{a}}_n$

$E\{\underline{\tilde{a}}_n^H \underline{\underline{R}}\underline{\tilde{a}}_n\} = \frac{m}{2} x_{min} Traza\{\underline{\underline{R}}\}$

$\aleph \equiv \frac{E\{x(n)\} - x_{min}}{x_{min}} 100\%$: exceso de error

$\aleph = \frac{m}{2} Traza\{\underline{\underline{R}}\}$

$m = \frac{2a}{P_x(n)}$

a : ruido de desajuste (i.e. $a = 0.1 \Rightarrow \aleph = 10\%$)

Conclusiones:

- Probar convergencia al óptimo
- Analizar la velocidad de convergencia
- En la fase de seguimiento, se ha de determinar el ruido de desajuste o lo que es lo mismo, su velocidad de seguimiento o capacidad de reacción a cambios.
- A altas velocidades de convergencia (o aprendizaje rápido) implican desajustes elevados.
- Hay un compromiso entre desajuste y velocidad de convergencia.

DSD y métodos de búsqueda aleatoria

DSD (Differential Steepest Descent), calcula el gradiente independientemente para cada coeficiente del filtro.

$$\nabla_n(q) = \frac{x_n(q,d) + x_n(q-d)}{2d}$$

$$\underline{a}_{n+1} = \underline{a}_n - \frac{m}{2}\underline{\nabla}_n$$

$$x_n(q,-d) = x_n - x_n' d + x_n'' \frac{d^2}{2}$$

$$x_n(q,+d) = x_n + x_n' d + x_n'' \frac{d^2}{2}$$

$$g = \frac{x_n'' d^2}{x_{min}} = \frac{d^2 r_{xx}(0)}{x_{min}}$$

$$g = \frac{d^2 Traza\{\underline{\underline{R}}\}}{Qx_{min}}$$

$$x_n(q,d) = \frac{1}{M}\sum_{n}^{n+M-1} |e(n)|^2$$

$$\aleph_{total} = \frac{3}{4}\frac{mQx_{min}}{Md^2} + \frac{d^2 Traza\{\underline{\underline{R}}\}}{Qx_{min}}$$

$$m = \frac{2a}{Traza\{\underline{\underline{R}}\}}$$

$$d_{opt}^2 = \sqrt{\frac{3}{2}\frac{Q^2 a x_{min}^2}{MTraza^2\{\underline{\underline{R}}\}}}$$

$$\aleph_{\text{total optimo}} = \sqrt{\frac{6a}{M}}$$

Ventajas

- Desajuste disminuye con la raiz cuadrada del número de términos empleados en la estimación del MSE.
- No es necesario el vector de datos

Desventaja:

- Se requieren 2MQ vectores por cada iteración.

Algoritmo LRS (Linear Random Search)

Para cada instance n,

- Generar un vector aleatorio con distribución uniforme en sus componentes $\underline{d}_n$.
- Actualizar los pesos como: $\underline{a}_{n+1} = \underline{a}_n + \frac{m}{2}\underline{d}_n$
- Calcular el nuevo error usando M vectores de datos x_{n+1}
- $\underline{a}_{n+1} = \begin{cases} \underline{a}_n & x_{n+1} > x_n \\ \underline{a}_{n+1} & \text{otherwise} \end{cases}$
- n++;

$$\aleph = \frac{mQs_d^2}{2M}$$

Algoritmo GRS (Guided Random Search)

Igual que LRS pero se diferencia en que cuando se encuentra una perturbación que hace disminuir el error, se avanza en esa dirección incrementado la m hasta que el error empeora y entonces $m = m_{básico}$ y se busca un vector de perturbaciones aleatorio ortogonal al presente.

Algoritmo RLS (Recursive Least Squares)

$$MSE(n) = bMSE(n-1) + (1-b)\,|\,e(n)\,|^2$$

$$\underline{\underline{R}}_{n+1} = b\,\underline{\underline{R}}_n + (1-b)\,\underline{X}_n \underline{X}_n^H$$

$$\underline{P}_{n+1} = b\,\underline{P} + (1-b)\,\underline{X}_n d^*(n)$$

$$\underline{a}_{n+1} = \underline{\underline{R}}_{n+1}^{-1} \underline{P}_{n+1}$$

$$M = \frac{1}{1-b} : \text{número de datos de la señal de entrada}$$

instante n: datos $\underline{a}_n, d(n), \underline{X}_n, \underline{\underline{R}}_n^{-1}$

- $y(n) = \underline{a}_n^H \underline{X}_n$
- $e(n) = d(n) - y(n)$
- $f = a\left(\underline{X}_n^H \underline{\underline{R}}_n^{-1} \underline{X}_n\right), a = \dfrac{1-b}{b}$
- $\underline{K}_n = \dfrac{a\,\underline{\underline{R}}_n^{-1} \underline{X}_n}{1+f}$: vectr ganancia
- $\underline{a}_{n+1} = \underline{a}_n + \underline{K}_n e^*(n)$
- $\underline{\underline{R}}_n^{-1} = \dfrac{1}{b}\underline{\underline{R}}_n^{-1} - \underline{K}_n \underline{K}_n^H \dfrac{1+f}{1-b}$
- $n++;$

Introduction

Disecrete-Time Sinusoidal

$x(n) = a\cos(wn + q), -\infty < n < +\infty$

$w = 2pf$

$x(n)$ periodic $\Leftrightarrow x(n) = x(N+n) \Leftrightarrow f \in Q$

$2pf_0 n = 2kp$

$f_0 = \dfrac{k}{N}$

$\cos(wn+q) = \cos((2pk + w)n + q) = \cos(w_k n + q), w_k = w + 2pk$

$\text{maximum oscillation} \Rightarrow w = \pm p, \text{f} = \pm\frac{1}{2} \Rightarrow x(n) = \{+1,-1,+1,-1,+1,\ldots\}$

Sampling of analog signals

$x(n) = x_a(nT)$

$t = nT = \dfrac{n}{F_m}$

$f = \dfrac{F}{F_m}$: normalized frequency

$w = \Omega T$

Continious-time signals	Discrete-time signals
$\Omega_{radians/\sec} = 2pF_{Hz}$	$w_{radians/sample} = 2pf_{cycles/sample}$
$-\infty < \Omega < +\infty$	$-p \le w \le +p$
$-\infty < F < +\infty$	$-\frac{1}{2} \le f \le +\frac{1}{2}$

Sampling Theorem

If the highest frequency contained in an analog signal $x_a(t)$ is $F_{max} = B$ and the signal is sampled at a rate $F_s > 2F_{max} \equiv 2B$, then $x_a(t)$ can be exactly recovered from its sample values.

Quantization

$x_q(n) = Q\{x(n)\}$

$e_q(n) = x_q(n) - x(n)$

$|e_q(n)| \le \dfrac{\Delta}{2}$

$\Delta = \dfrac{x_{max} - x_{min}}{L-1}$

Quantization of sinusoidal signals

$$\left.\begin{array}{l} P_q = \dfrac{\Delta^2}{12} \\ \Delta = \dfrac{2A}{2^b} \end{array}\right\} P_q = \frac{A^2/3}{2^{2b}} \Rightarrow SNQR = \frac{3}{2}2^{2b}, SNQR_{dB} = 1.76 + 6.02b$$

Discrete-Time Signals and Systems

Elementary signals

$d(n) \equiv \begin{cases} 1 & n = 0 \\ 0 & n \ne 0 \end{cases}$ unit sample sequence, unit impulse

$u(n) \equiv \begin{cases} 1 & n \ge 0 \\ 0 & n < 0 \end{cases}$ unit step signal

$u_r(n) \equiv \begin{cases} n & n \ge 0 \\ 0 & n < 0 \end{cases}$ unit ramp signal

$x(n) = a^n = (re^{jq})^n = r^n(\cos qn + j\sin qn)$ exponential signal

Classification of signals

- $E \equiv \sum_{n=-\infty}^{+\infty} |x(n)|^2 \quad 0 < E < \infty \Rightarrow x(n)$ is an energy signal
- $P \equiv \lim_{N\to+\infty} \frac{1}{2N+1} \sum_{n=-N}^{+N} |x(n)|^2 \quad 0 < P < \infty \Rightarrow x(n)$ is a power signal
- periodic signal $\Rightarrow$ power signal
- $x(n) = x(-n) \Rightarrow$ even signal; $x(n) = -x(-n) \Rightarrow$ odd signal; $x_e(n) = \frac{1}{2}(x(n) + x(-n))$; $x_o(n) = \frac{1}{2}(x(n) - x(-n))$

Discrete-time systems $x(n) \xrightarrow{\Gamma} y(n)$

- static: $y(n) = f\{x(n)\}$ memoryless
- dynamic: $y(n) = f\{x(n), x(n-1), \ldots\}$ with memory

- time variant: $x(n) \xrightarrow{\Gamma} y(n)$, $x(n-k) \xrightarrow{\Gamma} y(n-k)$
- linear: $\Gamma\{a_1x_1(n)+a_2x_2(n)\} = a_1\Gamma\{x_1(n)\}+a_2\Gamma\{x_2(n)\}$
- causal: output only depends on past and present.
- Stable: every bounded input produces a bounded output.

Analysys of Discrete-time systems Linear Time-Invariant

$$y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k)$$

$$y(n) = x(n) * h(n) = \sum_{k=-\infty}^{+\infty} h(k)x(n-k) = \sum_{k=-\infty}^{+\infty} x(k)h(n-k)$$

- Commutative law: $x(n)*h(n) = h(n)*x(n)$
- Associative law: $[x(n)*h_1(n)]*h_2(n) = x(n)*[h_1(n)*h_2(n)]$
- Distributive law: $x(n)*[h_1(n)+h_2(n)] = x(n)*h_1(n)+x(n)*h_2(n)$
- Causal: $h(n) = 0 \forall n<0$
- Stable: $\sum_{k=-\infty}^{+\infty} |h(n)| < \infty$
- FIR: $h(n) = 0 \forall n<0 \; and \; n \geq M$

Correlation of signals

$$y_{received}(n) = ax(n-D)+w(n)$$

$$r_{xy}(m) = \sum_{n=-\infty}^{+\infty} x(n)y(n-m) = \sum_{n=-\infty}^{+\infty} x(n+m)y(n) = r_{yx}(-m)$$

$$\sum_{n=-\infty}^{+\infty} |ax(n)+by(n-m)|^2 = a^2 r_{xx}(0) + b^2 r_{yy}(0) + 2ab r_{xy}(l)$$

$$|r_{xy}(m)| \leq \sqrt{E_x E_y} \qquad |r_{xx}(m)| \leq E_x$$

$$r_{xx}(m) = \frac{r_{xx}(m)}{r_{xx}(0)} : \text{normalized autocorrelation}$$

$$r_{xy}(m) = \frac{r_{xy}(m)}{\sqrt{r_{xx}(0) r_{yy}(0)}} : \text{normalized crosscorrelation}$$

Correlation of periodic signals

$$r_{xy}(m) = \lim_{M\to\infty} \frac{1}{2M+1} \sum_{n=-M}^{M} x(n)y(n-m) = \frac{1}{N}\sum_{n}^{N} x(n)y(n-m)$$

$$y(n) = x(n)+w(n) \Rightarrow r_{yy}(m) = r_{xx}(m)+r_{xw}(m)+r_{wx}(m)+r_{ww}(m)$$

Convolution realisation of correlation

$$r_{xy}(m) = x(m)*y(-m)$$
$$r_{xx}(m) = x(m)*x(-m)$$

Input-Output correlation sequences

$$y(n) = x(n)*y(n)$$

$$r_{yx}(m) = r_{xx}(m)*h(l)$$
$$r_{xy}(m) = r_{xx}(m)*h(-l)$$

$$r_{yy}(m) = r_{xx}(m)*r_{hh}(m) = r_{xx}(m)*h(m)*h(-m)$$

Frequency Analysis of signals and systems

Continious-Time signals

Jean Baptiste Joseph Fourier (1768-1830)

Periodic-Signals

Synthesis equations	$x(t)=\sum_{k=-\infty}^{+\infty} c_k e^{j2pkF_0 t}$ $x(t)=c_0+2\sum_{1}^{+\infty}\lvert c_k\rvert\cos(2pkF_0 t+\angle c_k)$ $x(t)=a_0+\sum_{1}^{+\infty}(a_k\cos(2pkF_0 t)+b_k sin(2p$
Analysis equations	$c_k=\frac{1}{T_p}\int_{T_p} x(t)e^{-j2pkF_0 t}dt$ $a_0=c_0$ $a_k=2\lvert c_k\rvert\cos\angle c_k$ $b_k=2\lvert c_k\rvert\sin\angle c_k$

Power Density Spectrum of Periodic Signals

$$P_x=\sum_{k=-\infty}^{+\infty}\lvert c_k\rvert^2=a_0^2+\frac{1}{2}\sum_{k=1}^{+\infty}\left(a_k^2+b_k^2\right)\qquad G_{xx}(f)=\sum_{k=-\infty}^{+\infty}\lvert c_k\rvert^2 d(f-F_0 k)$$

Aperiodic-Signals

Synthesis equation	$x(t)=\int_{-\infty}^{+\infty}X(f)e^{j2pft}df$
Analysis equations	$X(t)=\int_{-\infty}^{+\infty}x(t)e^{-j2pft}dt$

Power Density Spectrum of aperiodic Signals

$$S_{xx}(f)=\lvert X(f)\rvert^2\qquad E_{xx}=\int_{-\infty}^{+\infty}S_{xx}(f)df=\int_{-\infty}^{+\infty}\lvert x(t)\rvert^2 dt$$

Relationship between periodic and aperiodic transform

$$c_k=\frac{1}{T_p}X(kF_0)=\frac{1}{T_p}X(\frac{k}{T_p})$$

Discrete-Time signals

Periodic-Signals

Synthesis equations	$x(n)=\sum_{k=0}^{N-1}c_k e^{j\frac{2p}{N}kn}$
Analysis equations	$c_k=\frac{1}{N}\sum_{n=0}^{N-1}x(n)e^{-j\frac{2p}{N}kn}\qquad c_k=c_{k+N}$

$$P_x=\frac{1}{N}\sum_{n=0}^{N-1}\lvert x(n)\rvert^2=\sum_{n=0}^{N-1}\lvert c_k\rvert^2\qquad E_x=\sum_{n=0}^{N-1}\lvert x(n)\rvert^2=N\sum_{n=0}^{N-1}\lvert c_k\rvert^2$$

Aperiodic-Signals

Synthesis equations	$x(n)=\frac{1}{2p}\int_{2p}X(w)e^{jwn}dw$
Analysis equations	$X(w)=\sum_{n=-\infty}^{+\infty}x(n)e^{-jwn}$

$$E_x=\sum_{k=-\infty}^{+\infty}\lvert x(n)\rvert^2=\frac{1}{2p}\int_{-p}^{p}S_{xx}(w)dw\qquad S_{xx}(w)=\lvert X(w)\rvert^2\Rightarrow\begin{cases}S_{xx}(w)=S_{xx}(-w)\\ \angle X(w)=-\angle X(-w)\end{cases}$$

Relationship with Z-transform:

$$X(w)=Z\{x(n)\}\mid z=e^{jw}$$

Basic rectangular pulse tranform:

$$F\{u(n)-u(n-L)\}=Ae^{-j(w/2)(L-1)}\frac{sin(wL/2)}{sin(w/2)}$$

Properties of the Fourier Transform for Discrete-Time signals

$$a_1x_1(n)+a_2x_2(n)\overset{F}{\longleftrightarrow}a_1X_1(w)+a_2X_2(w)$$
$$x(n-k)\overset{F}{\longleftrightarrow}e^{-jwk}X(w)$$
$$x(-n)\overset{F}{\longleftrightarrow}X(-w)$$
$$x^*(n)\overset{F}{\longleftrightarrow}X^*(w^*)$$
$$nx(n)\overset{F}{\longleftrightarrow}j\frac{dX(w)}{dw}$$
$$x_1(n)*x_2(n)\overset{F}{\longleftrightarrow}X_1(w)X_2(w)$$
$$x_1(n)x_2(n)\overset{F}{\longleftrightarrow}\frac{1}{2p}\int_{-p}^{+p}X_1(l)X_2(w-l)dl$$
$$r_{x_1x_2}(m)=x_1(m)*x_2(-m)\overset{F}{\longleftrightarrow}X_1(w)X_2(-w)$$
$$r_{xx}(m)\overset{F}{\longleftrightarrow}S_{xx}(w)$$
$$e^{jw_0n}x(n)\overset{F}{\longleftrightarrow}X(w-w_0)$$
$$\cos(w_0n)x(n)\overset{F}{\longleftrightarrow}\frac{1}{2}\left(X(w-w_0)+X(w+w_0)\right)$$
$$\sin(w_0n)x(n)\overset{F}{\longleftrightarrow}\frac{1}{2j}\left(X(w-w_0)-X(w+w_0)\right)$$

Parseval's relation: $\sum_{n=-\infty}^{+\infty}x_1(n)x_2^*(n)=\frac{1}{2p}\int_{-p}^{+p}X_1(w)X_2^*(w)dw$

Some useful Fourier Transform Pairs

$$d(n)\overset{F}{\longleftrightarrow}1$$
$$x(n)=\begin{cases}1 & |n|\le L\\ 0 & |n|>L\end{cases}\overset{F}{\longleftrightarrow}\frac{\sin\left(L+\frac{1}{2}\right)w}{\sin\frac{w}{2}}$$
$$x(n)=u(n)a^n\overset{F}{\longleftrightarrow}\frac{1}{1-ae^{-jw}}$$
$$x(n)=\begin{cases}\frac{w_c}{p} & n=0\\ \frac{\sin w_c n}{pn} & n\neq 0\end{cases}\overset{F}{\longleftrightarrow}\prod\left(\frac{w}{2w_c}\right)$$

Frequency-Domain LTI Systems

$$H(w)=H(z)|_{z=e^{jw}}=\sum_{k=-\infty}^{+\infty}h(n)e^{-jwn}$$
$$Y(w)=X(w)H(w)$$
$$|Y(w)|=|X(w)||H(w)|$$
$$\angle Y(w)=\angle X(w)+\angle H(w)$$
$$S_{yy}(w)=S_{xx}(w)|H(w)|^2$$
$$H(w)=\frac{B(w)}{A(w)}=\frac{\sum_{k=0}^{M}b_ke^{-jwk}}{1+\sum_{k=1}^{N}a_ke^{-jwk}}=b_0\frac{\prod_{k=1}^{M}(1-z_ke^{-jwk})}{\prod_{k=1}^{N}(1-p_ke^{-jwk})}$$
$$|H(w)|^2=H(w)\overline{H}(w)$$
$$r_{yy}(m)=r_{xx}(m)*r_{hh}(m)\qquad S_{yy}(w)=S_{xx}(w)|H(w)|^2$$
$$r_{yx}(m)=r_{xx}(m)*h(m)\qquad S_{yx}(w)=S_{xx}(w)H(w)$$

Random Input signals

$$y(n)=x(n)*h(n)$$
$$m_y\equiv E\{y(n)\}=m_x\sum_{k=-\infty}^{+\infty}h(k)=m_xH(0)$$
$$g_{yy}(m)=E\{y^*(n)y(n+m)\}=\sum_{k=-\infty}^{+\infty}\sum_{j=-\infty}^{+\infty}h(k)h(j)g_{xx}(k-j+m)$$

(AWGN)
$$g_{xx}(m)=s_x^2d(m)\Rightarrow g_{yy}(m)=s_x^2\sum_{k=-\infty}^{+\infty}h(k)h(k+m)$$
$$g_{yy}(0)=P_y=s_x^2\sum_{k=-\infty}^{+\infty}|h(k)|^2=s_x^2\int_{-1/2}^{1/2}|H(f)|^2\,df$$
$$\Gamma_{yy}(w)=\sum_{m=-\infty}^{+\infty}g_{yy}(m)e^{-jwm}=\Gamma_{xx}(w)|H(w)|^2$$
$$\Gamma_{yx}(w)=\Gamma_{xx}(w)H(w)$$

Frequency-Selective Filters

$$h_{lp}(n)=\frac{\sin w_c pn}{pn} \qquad H_{hp}(w)=H_{lp}(w-p) \qquad h_{hp}(w)=(-1)^n h_{lp}(w)=\left(e^{jp}\right)^n h_{lp}(w)$$

Digital resonators: $p_{1,2}=re^{\pm jw_0}, r \text{ near} \le 1$

Notch filters: $z_{1,2}=e^{\pm jw_o}$

Comb filters: nulls occur periodically across the frequency band.

All-Pass filters: $|H(w)|=1\forall w$

Inverse Systems

$$y(n)=x(n)*h(n)$$
$$d(n)=h(n)*h_{INV}(n)$$

$$h_I(n)=0\forall n<0 \Rightarrow h_I(0)=\frac{1}{h(0)}$$

$$h_{INV}(n)=-\sum_{k=1}^{n}\frac{h(k)h_I(n-k)}{h(0)} \quad n\ge 1$$

- $|z_k|<1\forall k \Rightarrow \angle H(p)=\angle H(0)$ minimum phase system
- $|z_k|>1\forall k \Rightarrow \angle H(p)-\angle H(0)=Mp$ maximum phase system
- otherwise: mixed-phase system

$$H(z)\text{ minimum phase} \Rightarrow H^{-1}(z)\text{ stable}$$
$$H(z)\text{ stable} \Rightarrow H^{-1}(z)\text{ minimum phase}$$

$$H(z)\text{ non minimum phase}=H_{min}(z)H_{all-pass}(z)$$

$$H(z)=\frac{A(z)}{B(z)} \Rightarrow \begin{cases} H_{min}(z)=\dfrac{B_1(z)B_2(z^{-1})}{A(z)} \\ H_{all-pass}(z)=\dfrac{B_2(z)}{B_2(z^{-1})} \end{cases}, B(z)=B_1(z)B_2(z)$$

Deconvolution

(system identification)

$$y(n)=x(n)*h(n)\overset{casual}{=}\sum_{k=0}^{n}h(k)x(n-k)$$

$$h(0)=\frac{y(0)}{x(0)}$$

$$h(n)=\frac{y(n)-\sum_{k=0}^{n-1}h(k)x(n-k)}{x(0)}, n>0$$

$$S_{yx}(w)=H(w)S_{xx}(w)$$

$$H(w)=\frac{S_{yy}(w)}{S_{xx}(w)}=\frac{S_{yx}(w)}{|X(w)|^2}$$

Z-Transform

$$X(z)=\sum_{n=-\infty}^{+\infty}x(n)z^{-n}$$ Region of Convergence(ROC)

Finite-Duration Signals		***Infinite-Duration Signals***	
Causal:	$z\in C-\{0\}$	Causal:	$\lvert z\rvert>r_2$
Anticausal:	$z\in C-\{\infty\}$	Anticausal:	$\lvert z\rvert<r_1$
Two-sided:	$z\in C-\{0,\infty\}$	Two-sided:	$r_2<\lvert z\rvert<r_1$

Properties

$$a_1x_1(n)+a_2x_2(n)\overset{Z}{\longleftrightarrow}a_1X_1(z)+a_2X_2(z)$$

$$x(n-k)\overset{Z}{\longleftrightarrow}z^{-k}X(z)$$

$$a^nx(n)\overset{Z}{\longleftrightarrow}X(a^{-1}z)$$

$$x(-n)\overset{Z}{\longleftrightarrow}X(z^{-1})$$

$$x^*(n)\overset{Z}{\longleftrightarrow}X^*(z^*)$$

$$\mathrm{Re}\{x(n)\}\overset{Z}{\longleftrightarrow}\frac{1}{2}\left(X(z)+X^*(z^*)\right)$$

$$\mathrm{Im}\{x(n)\}\overset{Z}{\longleftrightarrow}\frac{1}{2}\left(X(z)-X^*(z^*)\right)$$

$$nx(n)\overset{Z}{\longleftrightarrow}-z\frac{dX(z)}{dz}$$

$$x_1(n)*x_2(n)\overset{Z}{\longleftrightarrow}X_1(z)X_2(z)$$

$$x_1(n)x_2(n)\overset{Z}{\leftrightarrow}\frac{1}{2pj}\oint_C X_1(v)X_2\left(\frac{z}{v}\right)v^{-1}dv$$

$$\text{if } x(n)\text{ causal}\Rightarrow x(0)=\lim_{z\to\infty}X(z)$$

Parseval's relation: $$\sum_{n=-\infty}^{+\infty}x_1(n)x_2^*(n)=\frac{1}{2pj}\oint_C X_1(v)X_2^*\left(\frac{z}{v}\right)v^{-1}dv$$

Basic Z-Transform Pairs

$$d(n)\overset{Z}{\longleftrightarrow}1$$

$$u(n)\overset{Z}{\longleftrightarrow}\frac{1}{1-z^{-1}}$$

$$a^nu(n)\overset{Z}{\longleftrightarrow}\frac{1}{1-az^{-1}}$$

$$na^nu(n)=\frac{az^{-1}}{\left(1-az^{-1}\right)^2}$$

Rational Z-Transforms

$$X(z)=\frac{N(z)}{D(z)}=\frac{\sum_{k=0}^{M}b_kz^{-k}}{\sum_{k=0}^{N}a_kz^{-k}}=\frac{b_0}{a_0}z^{-M+N}\frac{(z-z_1)\cdots(z-z_M)}{(z-p_1)\cdots(z-p_N)}=Gz^{N-M}\frac{\sum_{k=0}^{M}(z-z_k)}{\sum_{k=0}^{N}(z-p_k)}$$

$$y(n)=x(n)*h(n)\overset{Z}{\longleftrightarrow}Y(z)=X(z)H(z)$$

$$H(z)=\frac{Y(z)}{X(z)}=\sum_{n=-\infty}^{+\infty}h(n)z^{-n}$$

$$y(n)=-\sum_{k=1}^{N}a_ky(n-k)+\sum_{k=0}^{M}b_kx(n-k)\Rightarrow H(z)\frac{\sum_{k=0}^{M}b_kz^{-k}}{1+\sum_{k=1}^{N}a_kz^{-k}}$$

Analysis of LTI systems in the Z-domain

Any H(z)

causality: $\Leftrightarrow ROC\,\lvert z\rvert>z_1$

stability: $\Leftrightarrow ROC\in\lvert z\rvert=1$

Rational H(Z)

causality $\Leftrightarrow p_k$ most far away pole $\begin{cases}ROC\,\lvert z\rvert>\lvert p_k\rvert\\ \text{order}\{\text{num}\}\le\text{order}\{\text{denom}\}\end{cases}$

stability: $\Leftrightarrow\lvert p_k\rvert<1\,\forall k$

Unilateral Z-Transform

$$\mathrm{X}(z)=\sum_{n=0}^{+\infty}x(n)z^{-n}$$