How do you expect your wealth to behave under proportional betting?

It is a known fact that when someone is betting proportional to his wealth then the expected exponential growth rate per bet of this wealth is:

$$g(f) = \sum_i p_i \cdot \ln(1 + x_i \cdot f)$$

were p is the probability we will have a payoff x per unit bet, and f is our proportional betting fraction. Kelly criterion demands us to find the betting fraction f^* that maximizes g(f). This is equivalent as maximizing the expected logarithm of our wealth after a series of bets. The real-life question is what will be the distribution of our actual wealth after a sufficient number of trades if we follow the proportional betting strategy. It turns out that if B_N is our wealth after N bets and B_0 is our initial wealth then $\ln(B_N/B_0)$ follows a normal distribution N(A,B) with A=g(f)*N and B = sqrt(N)*sqrt[Sum_i {p_i*[ln(1+x_i*f)]^2} – [g(f)]^2]. So, our estimations about B_N are:

- Median (B_N) = B_0*exp(A)
- Mean (B_N) = B_0*exp(A+B^2/2)
  *Note that mean estimation may entail significant error and the exact solution is (1+f*E[x])^N
- Mode (B_N) = B_0*exp(A-B^2)
- Variance (B_N) = B_0*exp(2A+B^2)(exp(B^2)-1)

A sufficient value of N for the above approximations to be valid depends on the distribution of bet's payoffs. Skewed payoff distributions may cause erroneous estimations for mean and variance. A number of N=30 is usually sufficient in most cases and if the payoff distribution is smooth, even a number of 5 is sufficient.

Example

Let assume that we have a bet with payoff matrix per unit bet described by:

Payoff	*Probability*
0.13	80.00%
-0.05	10.00%
-0.50	10.00%

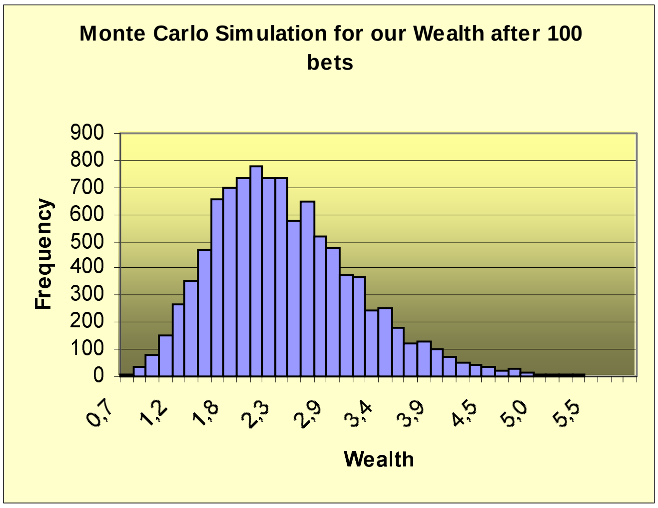
1. Substituting we have: g(f) = 0.80*ln(1+0.13*f) + 0.10*ln(1-0.05*f) + 0.10*ln(1-0.5*f)
2. We can easily maximize this function by changing f in excel using solver. We calculate this Kelly fraction to be f^* = 86.7%.
3. We choose to be conservative and bet a fraction of Kelly say 0.2*f^* that is around a 17% proportion of our wealth at each bet.
4. Substituting f = 0.17 into the formula gives us g(0.17) = 0.00775. This is the expected exponential growth rate of our wealth per bet.

5. If we want to find our wealth after say 100 consecutive proportional bets then we need to calculate the parameters A and B for N =100.
6. A = 0.00775 * 100 = 0.775
7. B = sqrt(100) * sqrt(0.80*(ln(1+0.13*0.17))^2 + 0.10*(ln(1-0.05*0.17))^2 + 0.10*(ln(1-0.5*0.17))^2 – 0.00775^2) = 0.33445

So we are now ready to derive the answers. If we assume our starting wealth to be B_0=1 then after 100 proportional bets with betting fraction 17%:

- Our Median Wealth will be **2.17**
- Our Mean Wealth will be **2.29**
- Our Mode Wealth will be **1.94**
- The Variance of our wealth will be **0.62**

A Monte Carlo simulation with 10000 trials of 100 consecutive bets gives us the following histogram for our wealth after 100 bets.



The estimations from the Monte Carlo simulation give us results very close to the theoretical ones above.

A final note about comparing trading systems

The above framework gives as a metric to compare different trading systems. Our goal is to choose the one that maximizes **the median** of our final bankroll after a specific **time frame**. If System_1 is producing N_1 trades per year and System_2 is producing N_2 trades per year then in order to compare them we do the following:

1. Calculate $N_1*g_1(f_1^*)$ and $N_2*g_2(f_2^*)$
2. If $N_1*g_1(f_1^*) > N_2*g_2(f_2^*)$ then System_1 is better, else System_2 is better.

Definitions

The *median* of a set of numbers arranged in order of magnitude is either the middle value or the arithmetic mean of the two middle values.

The *mode* of a set of numbers is that value which occurs with the greatest frequency.

Appendix A: Distribution of our wealth after N trades when payoffs x are uniformly distributed between a < 0 and b > 0

After N trades (N sufficiently large) $\ln(B_N/B_0)$ is normally distributed with:

- Mean = N*E[ln(1+xf)]
- Variance = N*Var[ln(1+xf)]

Solution

z=1+xf --> dz=fdx

E[ln(z)] = $\int_{1+af}^{1+bf} \frac{1}{f(b-a)} \ln(z)dz$ = 1/(f*(b-a)) * {(1+bf)ln(1+bf) – (1+af)ln(1+af) +(a-b)f}

E[(ln(z))^2] = $\int_{1+af}^{1+bf} \frac{1}{f(b-a)} (\ln(z))^2 dz$ = 1/(f*(b-a)) * {(1+bf)(ln(1+bf))^2 – 2(1+bf)ln(1+bf) + 2(1+bf) – (1-af)(ln(1+af))^2 + 2(1+af)ln(1+af) – 2(1+af)}

Var[ln(1+xf)] = E[(ln(z))^2] – { E[ln(z)]}^2

Example

a = -1, b = 2
E[ln(z)] = (1+2f)/(3f)*ln(1+2f) – (1-f)/(3f)*ln(1-f) – 1 --> f* = 0.716
We choose to be conservative and use f = 0.2*f* = 0.1432

Mean = N*0.06237
Variance = N*0.01387

After 20 trades:

A = Mean = 1.2474
B^2 = Variance = 0.2774

The estimations are:

Median Wealth = exp(A) = 3.48
Mean Wealth = exp(A + B^2/2) = 4.00. The exact solution is (1+f*E[x])^N = 3.99
Mode Wealth = exp(A – B^2) = 2.64
Variance Wealth = exp(2A + B^2) (exp(B^2) – 1) = 5.11
Standard Deviation Wealth = 2.26