On Distributions Associated with the Generalized Lévy’s Stochastic Area Formula

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Abstract

A closed-form expression is obtained for the conditional probability distribution of \( \int_0^t R^2_s \, ds \) given \( R_t \), where \((R_s, s \geq 0)\) is a Bessel process of dimension \( \delta > 0 \) started from 0, in terms of parabolic cylinder functions. This is done by inverting the following Laplace transform also known as the generalized Lévy’s stochastic area formula:

\[
E \left[ \exp \left( -\frac{\lambda^2}{2} \int_0^t R^2_s \, ds \right) \mid R_t = a \right] = \left( \frac{\lambda t}{\sinh(\lambda t)} \right)^{\delta/2} \exp \left( -\frac{a^2}{2t} \left( \lambda t \coth(\lambda t) - 1 \right) \right)
\]

We also examine the joint distribution of \((R^2_t, \int_0^t R^2_s \, ds)\).

Key words and phrases: Bessel process, density/distribution functions, parabolic cylinder functions, Laplace inversion.

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1. Introduction

1.1. If \((R_u, u \geq 0)\) is a Bessel process of dimension \( \delta > 0 \) started at 0, then the following formula is known to be valid (see e.g. [14]):

\[
E \left[ \exp \left( -\frac{\lambda^2}{2} \int_0^t R^2_s \, ds \right) \mid R_t = a \right] = \left( \frac{\lambda t}{\sinh(\lambda t)} \right)^{\delta/2} \exp \left( -\frac{a^2}{2t} \left( \lambda t \coth(\lambda t) - 1 \right) \right).
\]

If \( \delta = 1 \) and \( a = 0 \), then (1.1) leads to the distribution of the Brownian bridge \((b_s, s \geq 0)\) in the \(L^2\) norm which is identical to Smirnov’s distribution for his \(\omega^2\)-test. We recall below the relation between the integral of the square of the Brownian bridge and the supremum of the absolute value (see e.g. [3]):

\[
\int_0^1 b^2_s \, ds + \int_0^1 \tilde{b}^2_s \, ds \xrightarrow{\text{law}} \frac{4}{\pi^2} \sup_{0 \leq s \leq 1} |b_s|^2
\]

where \((\tilde{b}_s, 0 \leq s \leq 1)\) is an independent copy of \((b_s, 0 \leq s \leq 1)\).

If \( \delta = 2 \), then (1.1) is the Lévy’s stochastic area formula. Indeed, Lévy [10] showed that if \((X(t), Y(t))\) is an \(\mathbb{R}^2\)-valued Brownian motion, starting from \((0, 0)\), then for any \( \xi \in \mathbb{R} \) and \((x, y) \in \mathbb{R}^2\),

\[
E \left[ \exp \left( i \xi \int_0^t (X(u)dY(u) - Y(u)dX(u)) \right) \mid X(t) = x, Y(t) = y \right]
\]

\[
= \mathbb{E} \left[ \exp \left( -\frac{\xi^2}{2} \int_0^t R^2(u) \, du \right) \mid R(t) = r \right]
\]

\[
= \left( \frac{\xi t}{\sinh(\xi t)} \right) \exp \left( -\frac{r^2}{2t} \left( \xi t \coth(\xi t) - 1 \right) \right)
\]

where \( R^2 = X^2 + Y^2 \) and \( r^2 = x^2 + y^2 \).

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Lévy’s area formula arises naturally in some problems in analysis (explicit formula for the heat kernel corresponding to the Kohn-Laplacian of the Heisenberg group, see [7]), geometry (a probabilistic proof of the well-known index theorems of Atiyah and Singer due to J. M. Bismut, see [4, 5]) and statistical inference (parameter estimation and testing of statistical hypotheses for diffusion-type processes, see chapter 17 in [11]). We also note the close connection between the distributions of subordinated perpetuities and generalized Lévy’s formula for the stochastic area of planar Brownian motion (see [16] for details). For a historical account of Lévy’s area formula, we refer the interested reader to [9] and [13].

1.2. Equivalently, we can write the generalized Lévy’s stochastic area formula (1.1) as follows:

\[
\begin{align*}
E \left[ \exp \left( -u R_t^2 - v \int_0^t R_s^2 \, ds \right) \right] &= \left[ \cosh(\sqrt{2} v t) + \frac{2u}{\sqrt{2} v} \sinh(\sqrt{2} v t) \right]^{-\delta/2}.
\end{align*}
\]

In the Brownian case (i.e. \( \delta = 1 \)), the Laplace inversion of (1.6) has been undertaken by Abadir [1, 2] in 1995 who derived the joint density and distribution functions of the following two Brownian functionals:

\[
\frac{1}{2} (B_t^2 - 1) = \int_0^1 B_s \, dB_s \quad \text{and} \quad \int_0^1 B_s^2 \, ds
\]

where \( B_s \), \( 0 \leq s \leq 1 \) is a standard one-dimensional Brownian motion started at 0. These two functionals play an important role in unit root statistics (see [13]).

1.3. The paper is organized as follows. In Section 2 we derive explicitly the density of \( \int_0^t R_s^2 \, ds \) given \( R_t \) in terms of parabolic cylinder functions. In Section 3 we derive the joint density of \( (R_t^2, \int_0^t R_s^2 \, ds) \).

2. The density associated with the generalized Lévy’s area formula

The following theorem offers a method to invert (1.1); the result may be expressed in terms of parabolic cylinder functions.

**Theorem 2.1** The density \( f_{a,t} \) of \( \int_0^t R_s^2 \, ds \) given \( R_t = a \) is given by

\[
\begin{align*}
f_{a,t}(x) &= \frac{2^{\alpha} e^{\frac{\beta}{2}}}{\sqrt{2\pi}} e^{\frac{x}{2}} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} a^{2j} \sum_{k=0}^{\infty} \frac{\Gamma(j+\delta/2) k^{\delta/2-1} e^{-\frac{a^2}{4t} D_{2\nu+1}(\frac{a}{\sqrt{2t}})}}{k!} x^{\nu-1} \exp\left( -\frac{a^2}{2t} D_{2\nu+1}(\frac{a}{\sqrt{2t}}) \right) D_{2\nu+1}(\frac{a}{\sqrt{2t}})
\end{align*}
\]

where \( \alpha = 2k + \frac{\beta}{2} + 2j + \frac{\delta}{2} \), \( \beta = \frac{\delta}{2} + \frac{3}{2} \), \( D_\nu(x) \) is a parabolic cylinder function and \( (\nu)_k = \nu(\nu+1)...(\nu+k-1) = \Gamma(\nu+k)\Gamma(\nu) \) is the Pochhammer’s symbol.

**Proof:** First, according to [10; p. 259], we have

\[
\rho^e e^{-\rho x} = 2^{-\nu-\frac{\beta}{2} - \frac{\delta}{2} - 1} \exp\left( -\frac{a^2}{2t} D_{2\nu+1}(\frac{a}{\sqrt{2t}}) \right)
\]

Using the relation \( \coth(x) = 1 + 2(\exp(2x) - 1) \), then expanding the exponential:

\[
\begin{align*}
\left( \frac{\sqrt{2\lambda t}}{\sinh(\sqrt{2\lambda t})} \right)^{\delta/2} \exp\left( -\frac{a^2}{2t} \sqrt{2\lambda t} \coth(\sqrt{2\lambda t}) - 1 \right)
\end{align*}
\]

\[
= 2^{3\delta/4} e^{\beta/2} e^{x^2/2} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} a^{2j/2} \sum_{k=0}^{\infty} \frac{\Gamma(j+\delta/2) k^{\delta/2-1} e^{-\sqrt{2\lambda t} + 2\sqrt{2\lambda t} x + \frac{a^2}{4t}}}{(1 - e^{-2\sqrt{2\lambda t}})^{j+\delta/2}}
\]

\[
= 2^{3\delta/4} e^{\beta/2} e^{x^2/2} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} a^{2j/2} \sum_{k=0}^{\infty} \frac{\Gamma(j+\delta/2) k^{\delta/2-1} e^{-\sqrt{2\lambda t}}}{(1 - e^{-2\sqrt{2\lambda t}})^{j+\delta/2}}
\]

the termwise inversion of the series in (2.5) is readily justifiable by elementary estimates. □
Corollary 2.1 The density \( f_{0,t} \) of \( \int_0^t R_s^2 \, ds \) given \( R_t = 0 \) is given by

\[
\begin{align*}
\text{(2.6)} & \quad f_{0,t}(x) = \frac{2^k \pi^{k/2}}{\sqrt{2\pi}} \left( \frac{2}{\pi} \right)^{t/2} e^{-x^2/2} x^{-k/2} e \left( \frac{(k + \delta/4)^2 t^2}{x} \right) \quad D_{\frac{1}{4}+1} \left( \frac{2kt + \delta/2}{\sqrt{x}} \right) \\
\end{align*}
\]

where \( D_{\nu}(\xi) \) is a parabolic cylinder function and \( (\nu)_{\xi} \equiv \nu(\nu+1) \ldots (\nu+k-1) = \Gamma(\nu+k)/\Gamma(\nu) \) is the Pochhammer’s symbol.

Remark:

L. Tolmatz [15] determined the density (2.6) in the particular case for \( \delta = 1 \).

3. The joint density of \( (R_t^2, \int_0^t R_s^2 \, ds) \)

Theorem 3.1 The joint distribution \( g_{\nu}(R_t^2, \int_0^t R_s^2 \, ds) \) is given by

\[
\begin{align*}
\text{(3.1)} & \quad g_{\nu}(x,y) = \frac{1}{\sqrt{2\pi} \Gamma(\frac{1}{2})} \sum_{j=0}^{\infty} \left( \frac{2}{\pi} \right)^{t/2} e^{-x^2/2} x^{-j/2} y^{-j/2} \sum_{k=0}^{\infty} \frac{(j + \frac{\delta}{4})_k}{k!} D_{j+1} \left( \frac{2(k + j + \frac{\delta}{4}) t + \frac{\delta}{2}}{\sqrt{y}} \right) \\
\end{align*}
\]

where \( D_{\nu}(\xi) \) is a parabolic cylinder function and \( (\nu)_{\xi} \equiv \nu(\nu+1) \ldots (\nu+k-1) = \Gamma(\nu+k)/\Gamma(\nu) \) is the Pochhammer’s symbol.

Proof: Two methods lead to the same result (3.1). The first method follows from Theorem 2.1 by integrating the conditional density \( f_{\nu,t} \) with respect to the law of \( R_t^2 \)

\[
P(R_t^2 \in dx) = (2t)^{-\delta/2} \frac{1}{\Gamma(\frac{1}{2})} x^{\delta/2-1}e^{-x/2t} dx.
\]

This leads immediately to (3.1) and the details will be omitted. The second method is based on inverting the Laplace transform (1.6) and this can be done as follows.

Set \( X = R_t^2 \) and \( Y = \int_0^t R_s^2 \, ds \). Using formula (1.6), the joint density of \( X \) and \( Y \) is found to be given by

\[
g_{\nu}(x,y) = -\frac{1}{4\pi^2} \int_{-\infty}^{\gamma+\infty} \int_{-\infty}^{\gamma+\infty} e^{xu+yv} \left[ \cosh(\sqrt{2}vt) + \frac{2u}{\sqrt{2}v} \sinh(\sqrt{2}vt) \right]^{-\delta/2} \, du \, dv.
\]

We note that

\[
\begin{align*}
\left[ \cosh(\sqrt{2}vt) + \frac{2u}{\sqrt{2}v} \sinh(\sqrt{2}vt) \right]^{-\delta/2} &= \sum_{k=0}^{\infty} \frac{(-\frac{\delta}{4})_k}{k!} 2^{\frac{\delta}{2}} (\sqrt{2}v)^{\frac{\delta}{2}} e^{-\sqrt{2\nu}(2k + \frac{\delta}{4})} (2u - 2\sqrt{2}v)^k (2u + 2\sqrt{2}v)^{k+\frac{\delta}{2}} \\
\end{align*}
\]

Then, according to [10; p. 239], we have

\[
(3.3) \quad (p-a)_{\nu}(p+a)^{-\mu} = \frac{1}{\Gamma(\mu-\nu)} \mu^{\nu-1} e^{-a} I_1(-\nu; \mu - \nu; 2\nu) \quad \text{for} \quad R(\mu - \nu) > 0
\]

so it follows that

\[
\begin{align*}
\left[ \cosh(\sqrt{2}vt) + \frac{2u}{\sqrt{2}v} \sinh(\sqrt{2}vt) \right]^{-\delta/2} &= \int_0^\infty dx \, e^{-ax} \sum_{k=0}^{\infty} \frac{(-\frac{\delta}{4})_k}{k!} \frac{1}{\Gamma(\frac{1}{2})} (\sqrt{2}v)^{\frac{\delta}{2}} e^{-\sqrt{2\nu}(2k + \frac{\delta}{4})} x^{\frac{\delta}{2}-1} I_1(-k; \frac{\delta}{2}; x\sqrt{2}v). \\
\end{align*}
\]

By expanding Kummer’s function:

\[
(3.5) \quad _1F_1 \left( -k; \frac{\delta}{2}; x\sqrt{2}v \right) = \sum_{j=0}^{\infty} \frac{(-k)_j \left( \frac{\delta}{4} \right)_j}{j!} \left( \sqrt{2}v \right)^j
\]
we conclude as in the proof of Theorem 2.1:

\[
g_t(x, y) = \frac{1}{\sqrt{2\pi} \Gamma(\frac{\delta}{2})} \sum_{j=0}^{\infty} \frac{1}{j!} \left( \frac{\delta}{2} \right)_j (-i)^j \left( \frac{\delta}{2} \right)_i \left( \frac{\delta}{2} \right)_{i+j+1} e^{-\frac{1}{\sqrt{2}} \left( i + \frac{i}{2} t + \frac{\delta}{8} \right)^2 D_{\frac{\delta}{2}+j+1} \left( \frac{2(i + \frac{i}{2} t + \frac{\delta}{8})}{\sqrt{2}} \right)}.
\]

To show the equivalence between (3.6) and (3.1), let us compare the coefficients of these expressions. Since \((-i)_j = 0\) for \(i < j\) we see that the second summation in (3.6) takes place only over \(i \geq j\), so that by setting \(k = i - j\) the coefficients in (3.1) and (3.6) respectively become:

\[
C(j, k) = \frac{(-1)^j (j + \frac{i}{2})}{j!} \frac{1}{\Gamma(\frac{\delta}{2} + j + k)} \frac{1}{k!} \left( (-j)_j \right),
\]

\[
D(j, k) = \frac{1}{\Gamma(\frac{\delta}{2})} \frac{1}{j!} (\frac{\delta}{2})_j (\frac{\delta}{2})_{i+j} \left( (-j - k)_j \right).
\]

It is easily verified that:

\[
C(j, k) = \frac{(\frac{\delta}{2} + j + k)(-1)^i}{\Gamma(\frac{\delta}{2} + j)} \frac{1}{k!} = D(j, k).
\]

\[\square\]

Remarks:

1. A. Borodin kindly informed us that a similar expression for \(g_t\) appears in the new edition of [6] (see 1.9.8 p. 378).

2. Abadir [1] has derived the joint density of \((\sqrt{2} \int_0^t B_s \, dB_s, 2 \int_0^t B_s^2 \, ds)\) which correspond to the case \(\delta = 1\).

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References


