The Evolution of Geometry
and
Its Marriage to Physics

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Contents

As humans' understanding of geometries evolve, so does the understanding of the basic physical laws that govern the universe. Physics can be broken into several "specialized" areas, where the theories are approximately true. In classical mechanics, attributed mostly to Isaac Newton, speeds are much less than the speed of light. Special relativity, developed by Albert Einstein, applies when objects approach the speed of light, but space is flat and objects move at a constant velocity[1]. In general relativity, objects are allowed to approach the speed of light and have nonconstant velocities; space is curved.

Our understanding of these basic theories has been related to our understanding of the accompanying geometries.[2] Newton's work is based in the absolute flat space of Euclid. Einstein's special relativity is set upon a geometry developed by Hermann Minkowski. The general theory of relativity is based upon curved space developed by Carl Gauss and Georgi Riemann. It is the progression of understanding of geometry that permitted an accompanying change in physics. The very properties of the geometries of Euclid, Minkowski, and Gauss and Riemann dictate the properties of physical laws. The marriage of geometry and physics is essential for the progression of humans' understanding of physical laws..

[1]An object moving at a constant velocity moves at a constant speed in a straight line.

[2]It is important to note that although in many cases the geometry preceded the physics chronologically, that was not always the case. Einstein developed his special theory before Minkowski had a geometry that helped explain it. However, it was Minkowski's geometry that would help spur Einstein to his general theory. This paper presents events in a pedagogical, rather than chronological, order.

1 Laying the Ground Work

1.1 Greeks

The Greeks began by realizing that motions on Earth and in the sky can be modeled through mathematics. They understood that geometry, Greek for "earth measurement," can be used to not only model motion, but also to predict it.

Most Greek ideas came from the Babylonian culture; geometry is no different. However, it was the Greeks that realized a field of length 5 has the same properties as a house of length 5. This universal application of numbers allowed mathematics to be applied to nature. Thales was one of the prominent Greeks to visit Egypt and Babylon and bring back ideas in geometry. He used the Egyptian ideas, which included an early version of the Pythagorean Theorem, to begin to form the basis for Euclidean geometry centuries later. Thales recommended Egyptian study to a young pupil, Pythagoras.

While in Egypt, Pythagoras made great strides in the understanding of numbers. He was one of the first to classify numbers into even and odd. He also discovered many properties of triangular numbers[3]. Perhaps his most important contribution by Pythagoras and his society is the Pythagorean Theorem. In its simplest form, it states for any right triangle, the sum of the squares of the sides is equal to the square of the hypotenuse. It is often used as the test for flat

[3] Triangular numbers are numbers that can be formed by laying out triangles, i.e. the number ten is a triangle consisting of rows of 1, 2, 3, and 4

geometries.

1.2 Euclid

Euclid, one of the first major mathematicians, developed a geometry that would remain unchallenged for nearly 2000 years. His geometry was based in entirely pure thought, with no reference to the outside world.[6] His great work, *Elements*, first appeared around 350 B.C. This work was not original[4]; it simply was a compilation of the current thinking of the day regarding geometry.

In Euclid's *Elements*, five axioms are proposed. He uses these axioms to deduce the rules of his geometry. One important property of Euclid's geometry is based on a principle known later as Occam's razor. In its simplest terms, one should create a theory based on as "few ad hoc assumptions as possible."[6] The five axioms of Euclid represent these base assumptions. Later, Einstein would adopt this same premise - create a theory with as few assumptions as possible.

Euclid's geometry had no reference to time; he had no need for time. Until the fourteenth century, the concept of time was vague at best. Clocks would not appear until the 1330's, for people did not need to know time to any precision. Until then, time was expressed in relative units, with some degree of magnitude. One event was longer than another, but no number was identified.

[4] A scholar named Hippocrates wrote a book called *Elements* around 400 B.C.

2 Flat Geometry

There are several important consequences resulting from Euclid's geometry and his axioms. In the flat geometry of Euclid, triangles have angles that add up to 180 degrees, parallel lines never meet, and the shortest distance between two points is a straight line. These are all derivations from the five axioms, and they are used to define a flat geometry.

There are some physical consequences of this geometry. If a person walks along a line and another person walks along another line that is parallel, the two shall never meet. In order to go from one point to another in the shortest distance, one must walk in a straight line. Newton will come along and provide more rigorous rules concerning physics in Euclidean geometry.

In Euclidean geometry, the interval, or distance, between two points A and C is given by the Pythagorean Theorem. Suppose one knows the distance between A and B, a, and the distance between B and C, b. If a and b are perpendicular, then the interval between A and C, c, is given by:

$$c^2 = a^2 + b^2 \tag{1}$$

However, it was not until Rene Descartes applied Euclid geometry to his theory of numbers that physics would be able to be applied. Descartes was famous for his works with maps, graphical representations of the physical world. He is responsible for Cartesian coordinates, that is assigning values to different points in space.

By manipulating these coordinates, one could find the distance between any two points by using the distance formula, which is similar in form to the Pythagorean Theorem. If any two points in space are labeled by the Cartesian coordinates (x_1, y_1) and (x_2, y_2), then the distance bewtween them, d, is given by

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \quad (2)$$

The square of the distance, or interval, between two points is equal to the sum of two squares.

3 Physics of Flat Geometry

Descartes also made important contributions in physics. Using his model of geometry, he was able to completely describe the physics of a rainbow. He was able to present the refraction of light in its present form, know today as Snell's Law. In fact, his use of geometry to derive physics was so instrumental that he went on to say "my entire physics is nothing other than geometry." Descartes was one of the first to suggest a deep connection between geometry and physics.

Sir Isaac Newton was the first major theoretical physicist. Using the geometry known during his day, he derived three laws of motion that hold true in Euclid's geometry. He outlined these laws in his treatise, *Philosophiae Naturalis Principia Mathematica*[5] (*Mathematical Principles of Natural Philosophy*), published in 1687.

[5]Henceforth, it will be referred to as *Principia*

Similar to Euclid's axioms, Newton described two fundamental rules that concern his derivation of physics.

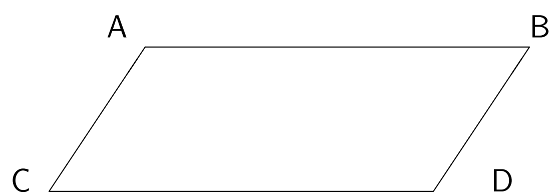
His two rules simply explain how he arrived at his conclusions; he observed the world around him. In his second law, he states that the same laws that hold true on Earth should hold true throughout the universe, just like "respiration is to man and beast; the descent of stones in Europe and *America*."[5] The laws of physics should hold for celestial and terrestrial mechanics. Einstein would later expound on this in his theory of relativity.

Newton's first law holds the deepest connection to geometry and holds the most significance. It states that an object moving at a constant velocity will continue to do so in a straight line until an outside force acts on it. Euclid clearly defines a straight line in his axioms for a flat geometry. However, in curved space, Euclid's axioms do not hold true. This will mean Newton's laws, especially the first law, can not hold true if Euclid's axioms do not hold true. If the geometry changes, then the laws of physics, as they are presented by Newton, must change.

In his second law, Newton emphasizes that only forces in the direction of motion, "the right line," affect the motion of an object. Forces that do not lie along the direction of motion will not change the acceleration, and therefore motion, of the object. Taken further, this implies that forces perpendicular to the motion of an object do not affect its motion.

3.1 Vector Addition

Newton's Corollary 1 from Principia[5] states "a body by two forces conjoined will describe the diagonal of a parallelogram, in the same time that it would describe the sides, by those forces apart." This corollary explains how force vectors are to be added.



It is this corollary that describes, in effect, vector addition that is taught in many introductory physics courses[6]. If there is one force that acts along $\overline{AB}$ and another force that acts along $\overline{AC}$, then the two forces will move the particle as described in the first law. It specifies that the two forces act independently of each other, as required by the seconds law, and the net result of the two forces will be a straight line from A to D. This is the diagonal of the parallelogram. It can be found using results from Euclidean geometry.

This result can only hold true in Euclid's flat geometry. If space is curved, then a parallelogram no longer has parallel lines, as defined in Euclid's fifth axiom. Therefore, Newton's description of vector addition can not hold true. However, to a good approximation, most terrestrial events can be described with Euclid's geometry, and Newton's results hold true.

[6] usually known as the graphical method or head to tail method of vector addition

3.2 Gravity

Newton's law of gravity is not as closely related to geometry as his laws of motion. The gravitational law simply states that the force of gravity between two objects is proportional to the mass of each object and inversely proportional to the square of the distance between them. However, a mathematical application of Newton's laws of motion to his law of gravitation reveals the path of two objects, such as the orbit of a planet around the sun, is a perfect ellipse in Euclid's geometry. This holds true to good precision; however, there are inaccuracies.

The biggest inaccuracy of Newton's law of gravitation involves the precession of Mercury's orbit. The point at which a planet is closest to the sun in its orbit is called the perihelion. According to Newton's theory, a planet will have the same perihelion every year. However, Jean-Joseph Leverrier announced in 1859 that he had observed that Mercury's orbit actually precesses, or moves, by a small amount. He found this amount to be 38 seconds per century.

This is left unexplained by Newton's law of gravitation; therefore, his law, which explains orbits in Euclid's geometry, does not fully describe Mercury. This means Euclid's geometry does not fully explain the heavens. It will be 250 years from Newton's gravitational theory before another theory emerges to fully explain this discrepancy.

4 Flat Geometry, With a Twist

4.1 Minkowski

As shown above, Euclid's geometry is unsatisfactory for the basis for physical laws throughout the universe. This begs the question: if not Euclidean geometry, then what? What geometries exist that might better provide the basis for physics?

H.G. Wells brought forth the idea of time as a fourth dimension in his book, *The Time Traveler*. In the novel, the time traveler details why time is needed:

> "You know of course that a mathematical line, a line of thickness nil, has no real existence. . . . Neither has a mathematical plane. These things are mere abstractions."
>
> "That's all right," said the Psychologist.
>
> "Nor, having only length, breadth, and thickness, can a cube have real existence."
>
> "There I object," said Filby. "Of course a solid body may exist. All real things -"
>
> " . . . But wait a moment. Can an instantaneous cube exist?"
>
> "Don't follow you," said Filby.
>
> "Can a cube that does not last for any time at all, have a real existence?"
>
> Filby became pensive. "Clearly," the Time Traveler proceeded, "any real body must have extension in four dimensions: it must have Length, Breadth, Thickness, and - Duration. . . . There are really four dimensions, three . . . of Space, and a fourth, Time. There is, however, a tendency to draw an unreal distinction between the former three dimensions and the latter because . . . our consciousness moves intermittently . . . along the latter from the beginning to the end of our lives."[3]

To further illustrate the need for four dimensions, imagine a party. In order to be invited to the party, one needs to know the intersection of streets where it is located and the floor the party is being held on. These are the three dimensions of space. However, one must also know the time of the party. This is the fourth dimension. Just as four dimensions are needed to arrive at the party, one needs four dimensions to uniquely describe events in physical space.[7] Space and time are intertwined necessarily.

One of the foremost mathematicians to suggest a geometry that included space and time was Hermann Minkowski. Einstein was a student of Minkowski's at Zurich Politechnikum in 1900. He professed Einstein to be a "lazy dog" in his early years. However, it was Minkowskian geometry that would allow Einstein to better formalize his general theory years later.

In his address to the 80th Assembly of German Natural Scientist and Physicians at Cologne in September 1908, Minkowski stated:

> The views of space and time which I wish to lay before you have sprung from the soil of experimental physics, and therein lies their strength. They are radical. Henceforth, space by itself, and time by itself, are doomed to fade away into mere shadows, and only a kind of union of the two will persevere an independent reality.

In Euclidean geometry, one uses points in space. Once space and time are linked, one no longer speaks of points, but of events. It is not enough to know where the party is, a point in space, one must also must also know when the party is, an event in spacetime.

In Minkowski's geometry, two events are separated by an interval, $\Delta\tau$, that is analogous to the distance between two points in Euclidean geometry. The distance in Euclidean geometry, where space and time are treated separately, is simply

[7] This explanation is based on a similar scenario by George Gamow, a Russian physicist.

given by the Pythagorean Theorem (see equation 1). However, in Minkowskian geometry, where space and time are intimately related, the interval is given by:

$$\Delta\tau^2 = \mathrm{time}^2 - \mathrm{space}^2 \tag{3}$$

Here the interval, or distance between two events, is comprised of a time quantity squared MINUS a space quantity squared. In Euclid, the interval is the sum of the squares. Minkowski specified that the interval is the same for all observers; it also includes a dimension of time. Here is where Minkowskian geometry differs from Euclid, for this allows complex numbers to enter the fray. If the space quantity is larger than the time quantity [8], then the interval involves the square root of a negative number. This is not possible in Euclidean geometry.

4.2 Einstein and the Special Theory

The special theory of relativity holds for observers that move with constant velocities that approach the speed of light. Einstein actually had the mathematics for the special theory of relativity before Minkowski developed the geometry for it. Einstein used two basic postulates, or axioms, to deduce many new ideas about the laws of physics.

Relativity draws its name from the first postulate. In layman terms, the laws of physics are the same for all observers moving at a constant velocity. Rela-

[8] this is known as a space-like interval

tive observers at a constant velocity observe events that follow the same laws of physics; the laws of physics are invariant to all observers. This is also evident in Minkowskian geometry. The interval, or distance between events, is the same for all observers; it is invariant. However, for Einstein, the distance is not any different from any other physical quantity, so it should be invariant, as specified by the first postulate. Einstein would not allow "physics to be reduced to geometry." However, he would accept the relationship of geometry to physics and use curved geometries to better formulate his general theory.

4.3 Consequences of the Special Theory

Since in special relativity space and time are related, this new theory produces major implications. As an object approaches the speed of light, it is contracted along its direction of motion. Since length (space) is affected by motion, it should hold that time is as well. This is known as time dilation. Time moves slower the faster an object moves.

In length contraction, also known as Lorentz contraction, objects appear shorter when measured by a person that observes the object to be moving than when measured by a person who observes the object to be at rest. For a person at rest, a ruler is measured to be 12 inches. However, if the ruler is moving just over half of the speed of light[9], the observer will measure the ruler to be much shorter,

[9]The speed of light in an inertial reference frame is 186,000 miles per second. To move at half this speed is no small task.

approximately 10 inches. This Lorentz contraction is given by a formula Lorentz first derived. For an an object of rest length L_o, the measured length of the object, L will be given by

$$L = L_o\sqrt{1 - v^2} \tag{4}$$

where v, the speed of the observer, is expressed as a fraction of the speed of light.

For time dilation, clocks move slower the faster they move. The faster an object moves, the slower time will pass for that object. This has been confirmed with atomic clocks aboard passenger jets. Atomic clocks, which keep time to great precision are set to the same time. One clock is flown aboard a jet, and then returned to the ground. The other clock is kept stationary on the ground. When the times measured by each clock is compared, the clock that was aboard the jet has recorded less time elapsed. For a person, this implies that a person will age slower. In fact, this idea is used in the movie *Contact*. In the movie, a person stays in orbit above earth in hopes that he will age slower.

Einstein's special theory has been verified by several experiments. The most notable experiment involve the observation of muons. Muons, a by-product of high energy collisions in space, normally have a lifetime as measured at rest to be of the order of 2.198 microseconds. The journey should take several seconds from outer space to the instruments on Earth, yet their lifetimes at rest are much shorter. This indicates that the muons should not survive the journey. The only explanation is that the muons, which travel near the speed of light, experience a dilated lifetime, due to their speed. The very observation of muons provide confirmation of the special theory of relativity.

4.4 Gauss and Riemann

Euclid created a mathematical structure based on the geometry of space; it was quantified by Descartes. However, it has been shown that this is not a complete description of space. It was Carl Gauss that first realized that Euclid's parallel postulate (his fifth axiom) did not hold true in all geometries, that is there exists geometries in which parallel lines do intersect. He realized that by changing the fifth axiom, he could describe new, Non-Euclidean geometries.[1] Gauss worked out today what is known as hyperbolic geometry. Gauss' most important contribution to the evolution of geometry, however, was the idea that a surface can curve without being curved into anything. In more technical terms, one can study a curved surface without reference to a higher dimensional Euclidean geometry.

Georgi Riemann, like Gauss, developed a Non-Euclidean space. He never mentioned this Non-Euclidean space by name; however, his work provided the basis for representing spheres in two dimensional elliptical space. He used differential geometry, the process of using small regions of a larger surface. This method of approximating complex surfaces as regions of infinitesimally small, simpler ones will be instrumental in advances in physics, especially in Einstein's general theory of relativity.

5 Curved Geometry

In curved space, the properties that are deduced from Euclid's axioms no longer hold true. For instance, the angles of triangle no longer add up to 180 degrees. In fact, in Gaussian hyperbolic space, it will add up to something less than 180 degrees; how much it differs by is a number called the angular defect. Henri Poincare adapted Gauss' hyperbolic space into concrete entities. He defined lines in this curved space to be geodesics, the shortest path between two points. In Euclidean geometry, the shortest path between two points is defined to be a straight line.

In Riemann's model, Euclid's axiom of lines extending indefinitely in either direction is false. The equator along Earth is a prime example. The line has definite length; it is simply 2π times the radius of the circle. Also gone is the idea of betweenness, for there is more than one way to go from one point to another. Again, use the globe as an example. For a person in New Orleans, he can arrive in New York by flying north in a straight line as defined in curved space, or he can travel south and fly around the South Pole. Maybe the most important axiom that is refuted is the parallel postulate. Parallel lines, as defined in Euclidean geometry, no longer do not intersect. In curved space, two objects that were thought to never cross in Euclidean space will meet, according to the properties of the curved geometry.

6 Physics of Curved Geometry

Euclidean flat geometry is fundamental to Newton's laws of motion. However, in curved space, these laws have to be modified. When new theories of physics were being developed with curved space, the goal was to keep as much of Newtonian mechanics as possible. For instance, Newton's first law of motion specify objects move in a straight line in the absence of outside forces. In curved space, these objects move in geodesics.

Imagine an ant walking along an apple; it is unaware of the surface. As the ant walks near the stalk of the apple, it will follow a path defined by a geodesic. This path will draw the ant nearer the stalk. The ant, which does not know the surface on which it is walking is curved, observes the stalk drawing nearer. It deduces that since he is being drawn toward the stalk, there must be some force, in accordance to Newton's first law, that is making him deviate from a straight line path. However, the detour from a straight line is simply the product of the surface on which it is walking; the detour is not due to a force.[10]

[10] This scenario is adapted from John Wheeler.

6.1 Einstein and the General Theory

The Minkowski theory of geometry helped unify geometry and physics. In fact, Einstein once stated his discovery of general relativity had been "greatly facilitated" by the geometry of Minkowski, as applied to special relativity. It is interesting to note that in situations where gravity is weak and objects move much slower than the speed of light, Einstein's general theory reduces to Newton's well known law of gravitation.

The general theory of relativity applies in situations in which an object moves near the speed of light and gravitational fields are strong. The general theory is based on the Equivalence Principle, which states an observer does not know if effects from acceleration, i.e. a force, are due to the acceleration itself or if it is a product of a local geometry. Formally, the Equivalence Principle states

> . . . we shall therefore assume the complete physical equivalence of a gravitational field and the corresponding acceleration of a reference frame. This assumption extends the principle of relativity to the case of uniformly accelerated motion of the reference frame.

If the ant was unaware of the surface of the apple and still used Newtonian mechanics and Euclidean geometry, it would probably state the force causing it to deviate from a straight line path is gravity. Until the general theory, many would agree. Einstein's general theory simply states that the perceived force acting on the ant is simply due to walking along a curved surface; the force is an unnecessary addition to the theory.[11] This "force" is simply due to the geometry of space (and

[11]Remember, Occam's razor - one should eliminate as many unnecessary things from a

necessarily time).

When one looks at the surface of an apple through a magnifying glass, the surface looks approximately flat. This is an essential idea to the general theory-small parts of a surface are flat, to an approximation. The projection of the apple's curved surface looks flat when viewed through a magnifying glass. Riemann's differential geometry is able to be applied here. A complex surface can be thought of as many smaller, simpler surfaces. Here, the use of the magnifying glass reduces a complex three dimensional surface of an apple to a simpler two dimensional one. It was Einstein who realized that Euclidean geometry only held locally, and he needed a curved geometry to fully develop his gravitation theories. He stated if the Equivalence Principle is true, then "Euclidean geometry cannot hold." [5]

Using the Minkowski development of special relativity and the geometry thereof, Einstein derived equations to describe the curved geometry of different universes. The equations prove to be non-linear and are extremely difficult to solve. However, in limiting situations, there are shown to reduce to well known equations from Newtonian mechanics and special relativity. The equations involve a metric tensor[12], which helps describe the geometry of the space. The equations also describe the curvature of the surface through the Riemann curvature tensor. These tensors, along with a tensor that describes the distribution of mass and energy,

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theory as possible.

[12]A tensor is a mathematical tool that helps analyze four dimensional spacetime using mathematics.

fully explain gravitation and other phenomena.

6.2 Consequences of the General Theory

General relativity dictates that space and time must be curved in order to accurately describe space. So far, only the effects on space have been discussed. One effect on time is the gravitational red-shift. Einstein's theory predicts this small, but significant shift. For a radio station that broadcasts at 1070 kHz (AM) on top of a ten story tower, a person on the ground would need to tune into 1070.00000000003 kHz. Although small for everyday needs, this has major implications at a somewhat larger scale.

More relevant application of Einstein's general theory include Global Positioning Satellites (GPS). In order for the GPS systems to stay in sync, one must take into account deviations as dictated by the general theory. Newton's theory would not account for this abberation. Einstein's equations fully describe the bending of light by gravity; without this, GPS would not function correctly.

One of the early tests (and victories) of the general theory is gravitational lensing. Due to gravity, light is bent, similar to how a lens bends and focuses light. In 1919, an expedition led by Arthur Eddington set out to Brazil to observe a solar eclipse. During night time, one observes the position of the stars. During a solar eclipse, if one observes the position of the stars when the sun is present, then the light from the stars will be deflected by the sun; it will appear as if the stars were shifted. Months after the eclipse, after the data was processed, Einstein's theory was proven to accurately predict the deflection from the sun. This implied that gravity bends light, which is massless, in accordance to Einstein's theory of gravitation, not Newton's.

Orbital precession marked another triumph for the general theory. As remarked earlier, Mercury's orbit is observed to precess. By 1882, more accurate measurements showed the precession to be 43 seconds per century. On November 15, 1915, Einstein published a paper that precisely predicted the precession of Mercury's orbit. After several corrections and recalculations, Einstein published his final version of the general theory of relativity November 25, 1915. After a long progress that involved several errors along the way, Einstein would write of himself, "That fellow Einstein suits his convenience. Every year he retracts what he wrote the year before."

An extension of general relativity involves holograms. A hologram provides a two dimensional surface that houses a three dimensional image. One can move around the hologram. This is similar to the reduction of the three dimensions of the apple to two dimensions of the magnifying glass in the previous example. If everyday observations are in four dimensions, then how is it proven that this is not simply a holographic image? Are there more than four dimensions for space time? In general relativity, our observations are simply a product of four dimensional geometry. Can our observations be holograms of a higher dimensional geometry?

These questions are left unanswered; however, current theories are being developed (namely string theory and M-theory) that attempt to find the geometry to help answer these physics questions.

7 Conclusions

Euclidean geometry does not hold as an accurate model of the model universe. However, the Euclidean method provided the basis for the "axiomatic method."[1] This is a theme that recurs throughout geometry and physics. It a style of reasoning that appears from Newton to Einstein, from Euclid to Riemann. It is this style that allows major theories to be built around Occam's razor; based on a few axioms that are assumed to be correct, many ideas can be developed. As new geometries are developed, so does the accompanying physics. As geometry evolved from flat, with an absolute space, to flat space time, and then to curved space, so did our physics, from Newtonian mechanics to the special and general theories of relativity. The rules of physics have evolved to encompass more phenomena. However, the evolution of physics can not be so without the necessary evolution of geometry. Physics and geometry are bound to a lifetime together, a marriage of necessity.

8 Bibliography

References

[1] John Barrow. *The Book of Nothing; Vacuums, Voids, and the Latest Ideas about the Origins of the Universe*. 2000.

[2] Albert Einstein. On the electrodynamics of moving bodies. 1905.

[3] J. Richard Gott. *Time Travel in Einstein's Universe; The Physical Possibilites of Travel Through Time*. 2001.

[4] Stephen Hawking. *The Universe in a Nutshell*. 2001.

[5] Stephen Hawking. *On the Sholders of Giants; The Great Works of Physics and Astronomy*. 2002.

[6] Leonard Mlodinow. *Euclid's Window; The Story of Geometry from Parallel Lines to Hyperspace*. 2001.

Table 1: Euclid's Axioms, in modern form

Axiom 1 Given any two endpoints, a line segment can be drawn with those points as its endpoints.

Axiom 2 Any line segment can be extended indefinitely in either direction.

Axiom 3 Given any point, a circle with any radius can be drawn with that point at its center.

Axiom 4 All right angles are equal.

Axiom 5 Given a line and an external point (a point not on the line), there is exactly one other line (in the same plane) that passes through the external point and is parallel to the line.[6]

Table 2: Newton's Two Rules of Reasoning

Rule 1 We are to admit no more causes of natural things than such as are both true and sufficient to explain their causes.

Rule 2 Therefore to the same natural effects we must, as far as possible, assign the same causes.

Table 3: Newton's Three Laws of Motions

First Law Every body perseveres in its state of rest, or of uniform motion in a right line, unless it is compelled to change that state by forces impressed thereon.

Second Law The acceleration of motion is ever proportional to the motive force impressed; and is made in the direction of the right line in which that force is impressed.

Third Law To every action there is always opposed an equal reaction; or the mutual actions of two bodies upon each other are always equal, and directed to contrary parts.[5]

Table 4: Einstein's Two Postulates in Special Relativity

Postulate 1 The laws by which the states of physical systems undergo change are not affected, whether these changes of state be referred to the one or the other of two systems of coordinates in uniform translatory motion.

Postulate 2 Any ray of light moves in the "stationary" system in accordance with the determined velocity c, whether the ray be emitted by stationary or by a moving body.[5]