Assignment 3

Total: 45

Chapter 4

3) From Eqt. (20)

$$\begin{cases} -\omega^2 M_1 u = Cv\,[1 + \exp(-iKa)] - 2Cu \\ -\omega^2 M_2 v = Cu\,[\exp(iKa) + 1] - 2Cv \end{cases}$$

At $K = \frac{\pi}{a}$,

$$\begin{cases} -\omega^2 M_1 u = -2Cu \quad \text{——— (1)} \quad 1 \\ -\omega^2 M_2 v = -2Cv \quad \text{——— (2)} \quad 1 \end{cases}$$

$\Rightarrow$ we cannot find $\frac{u}{v}$ 1

From (1), $\omega^2 = \frac{2C}{M_1}$

Sub. into (2), $-\left(\frac{2C}{M_1}\right)M_2 v = -2Cv$ 1

If $v \neq 0$, then $M_1 = M_2$. But we assumed that $M_1 \neq M_2$, hence $v = 0$, and the motion is only in the planes whose motion is described by $u$. The opposite result holds for the other $\omega$, [i.e. $\omega^2 = \frac{2C}{M_2}$]. 1

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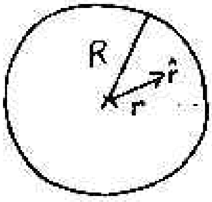
6a) In Gaussian units, the electric field in the interior of a uniformly charged sphere is

$$\vec{E} = \frac{-er}{R^3}\,\hat{r} \quad 1$$




$$\vec{F} = \frac{-e^2 r}{R^3}\,\hat{r} \quad 1$$

For simple harmonic motion, $F = -kx$ 1

$$\therefore \quad k = \frac{e^2}{R^3}$$

$$\omega = \underset{1}{\sqrt{\frac{k}{M}}} = \sqrt{\frac{e^2/R^3}{M}} = \underset{1}{\sqrt{\frac{e^2}{MR^3}}} \quad ※$$

b) For sodium, $m \simeq 4 \times 10^{-23}$ g and $R \simeq 2 \times 10^{-8}$ cm

$$\omega = \sqrt{\frac{e^2}{MR^3}} = \sqrt{\frac{(4.80325 \times 10^{-10})^2}{4 \times 10^{-23} \times (2 \times 10^{-8})^3}}\ s^{-1} \quad 1$$

$$= 2.7 \times 10^{13}\ s^{-1} \quad ※ \quad 1$$

$$\simeq 3 \times 10^{13}\ s^{-1}$$

c) The max. phonon wavevector is of the order of $1/R = 10^{8}\ cm^{-1}$. Assume that the dispersion relation is linear roughly to ① this order in k, and

the speed of sound $v = \frac{\omega}{k_{max}} \simeq \frac{2.7\times10^{13}}{10^{8}} = 2.7\times10^{5}\ cm\ s^{-1}$ ①
$\simeq 3000\ m\ s^{-1}$ #
①

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Chapter 5

4a) $C_V = \frac{\partial U}{\partial T}\Big|_V$ ①

Consider the thermal energy $U$

$$U = \int d\omega\ D(\omega) \cdot \langle n(\omega) \rangle\ \hbar\omega$$ [ Eqt (25) ]

$$= \int_0^{\omega_D} d\omega\ D(\omega) \frac{\hbar\omega}{e^{\hbar\omega/\tau} - 1}$$ ①

[ using Debye Model : Planck distribution for $\langle n(\omega) \rangle$ ]

Consider the density of states $D(\omega)$

$$D(\omega) = \frac{dN}{d\omega}$$ ①

$$= \frac{d}{d\omega}\left[\frac{\pi k^2}{(2\pi/L)^2}\right]$$ ①

$$= \frac{d}{d\omega}\left[\frac{L^2}{4\pi}\left(\frac{\omega}{v}\right)^2\right]$$ [ $k = \frac{\omega}{v}$ in Debye Model ]

$$= \frac{L^2\omega}{2\pi v^2}$$

$$\therefore\ U = \int_0^{\omega_D} \frac{L^2\omega}{2\pi v^2} \cdot \frac{\hbar\omega}{e^{\hbar\omega/\tau} - 1}\, d\omega$$

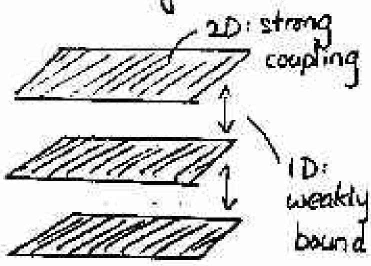
Let $x = \frac{\hbar\omega}{\tau}$, $dx = \frac{\hbar}{\tau}\, d\omega$, $x_D = \frac{\hbar\omega_D}{\tau}$

$$U = \frac{L^2}{2\pi v^2} \int_0^{x_D} \frac{\tau^3 x^2}{\hbar^2 (e^x - 1)}\, dx$$

$$= \frac{L^2 (k_B T)^3}{2\hbar^2 \pi v^2} \int_0^{x_D} \frac{x^2}{e^x - 1}\, dx$$ ① [ $\tau = k_B T$ ]

$$C_V = \frac{\partial U}{\partial T} = \frac{3L^2 k_B^3\ T^2}{2\hbar^2 \pi v^2} \int_0^{x_D} \frac{x^2}{e^x - 1}\, dx \propto T^2$$ #
① ①

4b). Apart from 2-D vibration in the plane, the system behaves as a linear structure with each plane as a vibrating unit.




$U = U(2D) + U(1D)$ (1)

$C_V = \alpha T^2 + \beta T$ (1) at low temp.

At extremely low temperature, $C_V \propto T$ (1) #

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Chapter 6

1) K.E. + P.E = Total energy

$$= \int_0^\infty \varepsilon f(\varepsilon) D(\varepsilon) d\varepsilon \quad (1)$$

At 0K, P.E. = 0, $f(\varepsilon) = 1$ (1)

$$\therefore K.E = \int_0^{\varepsilon_F} \varepsilon D(\omega) d\varepsilon$$

$$= \int_0^{\varepsilon_F} \varepsilon \cdot \frac{V}{2\pi^2} \cdot \left(\frac{2m}{\hbar^2}\right)^{3/2} \varepsilon^{1/2} d\varepsilon \quad (1) \quad [\text{Eqt. } (20)]$$

$$= \frac{V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} \int_0^{\varepsilon_F} \varepsilon^{3/2} d\varepsilon$$

$$= \frac{V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} \cdot \frac{2}{5} \varepsilon_F^{5/2}$$

$$= \frac{V}{\pi^2}\left(\frac{2m\varepsilon_F}{\hbar^2}\right)^{3/2} \cdot \frac{1}{5} \varepsilon_F$$

$$\because \varepsilon_F = \frac{\hbar^2}{2m}\left(\frac{3\pi^2 N}{V}\right)^{2/3} \quad [\text{Eqt. } (17)]$$

$$\left(\frac{2m\varepsilon_F}{\hbar^2}\right)^{3/2} \cdot \frac{V}{\pi^2} = 3N \quad (1)$$

$$\therefore KE = 3N \cdot \frac{1}{5}\varepsilon_F \quad (1)$$

$$= \frac{3}{5} N \varepsilon_F$$ #

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[In the quiz, point marks will be doubled for this question]

3). For electron, in 2-D,

$D(k)\,d^2k = (\text{spin})(\text{states per k-area})(\text{k-area element})$

$$= 2 \times \left(\frac{L}{2\pi}\right)^2 \times 2\pi k\, dk \quad ①$$

$$= \frac{L^2}{\pi} k\, dk$$

$$\because \varepsilon = \frac{\hbar^2 k^2}{2m} \Rightarrow d\varepsilon = \frac{\hbar^2}{m} k\, dk \quad ①$$

$$\therefore D(\varepsilon)\,d\varepsilon = D(k)\,d^2k = \frac{L^2}{\pi}\frac{m}{\hbar^2}\,d\varepsilon \quad ①$$

Density of states per unit area $D'(\varepsilon) = \frac{m}{\pi\hbar^2}$

The total no. of electrons per unit area

$$n = \int_0^\infty f(\varepsilon) D'(\varepsilon)\,d\varepsilon \quad ①$$

$$= \frac{m}{\pi\hbar^2}\int_0^\infty \frac{d\varepsilon}{e^{(\varepsilon-\mu)/k_BT}+1} \quad ① \qquad [\text{Eqt. } (5)]$$

Let $x = e^{(\varepsilon-\mu)/k_BT}$, $dx = \frac{x}{k_BT}\,d\varepsilon$

$$\therefore n = \frac{m}{\pi\hbar^2}(k_BT)\int_{e^{-\mu/k_BT}}^{\infty} \frac{1}{x(x+1)}\,dx$$

$$= \frac{mk_BT}{\pi\hbar^2}\left[-\ln\frac{1+x}{x}\right]_{e^{-\mu/k_BT}}^{\infty}$$

$$= \frac{mk_BT}{\pi\hbar^2}\ln\frac{x}{1+x}\Big|_{e^{-\mu/k_BT}}^{\infty}$$

$$= \frac{mk_BT}{\pi\hbar^2}\left[\ln 1 - \ln\frac{e^{-\mu/k_BT}}{1+e^{-\mu/k_BT}}\right] \qquad ③$$

$$= \frac{mk_BT}{\pi\hbar^2}\ln\frac{1+e^{-\mu/k_BT}}{e^{-\mu/k_BT}}$$

$$e^{\frac{n\pi\hbar^2}{mk_BT}} = \frac{1+e^{-\mu/k_BT}}{e^{-\mu/k_BT}}$$

$$e^{-\mu/k_BT}\left(e^{\frac{n\pi\hbar^2}{mk_BT}} - 1\right) = 1$$

$$\frac{-\mu}{k_BT} = \ln\frac{1}{e^{n\pi\hbar^2/mk_BT}-1}$$

$$\mu = k_BT\ln\left[\exp\left(\frac{n\pi\hbar^2}{mk_BT}\right)-1\right] \quad ①$$

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[ Note: low temperature : $\mu = k_B T \ln [e^{n\pi\hbar^2/mk_BT} - 1]$

$$\simeq k_B T \frac{n\pi\hbar^2}{mk_BT}$$

$$= \frac{n\pi\hbar^2}{m} = \varepsilon_F$$

high temperature : $\mu = k_B T \ln [\frac{n\pi\hbar^2}{mk_BT}] = k_B T \ln \frac{T_F}{T}$ ]

5). From Eqt. (17)

$$\varepsilon_F = \frac{\hbar^2}{2m} \left(\frac{3\pi^2 N}{V}\right)^{2/3} \qquad 1$$

Given density = 0.081 g cm$^{-1}$

$$\frac{N}{V} = 0.081 \times \frac{6.022 \times 10^{23}}{3} \text{ atoms cm}^{-3} \qquad 1$$

$$= 1.62594 \times 10^{22} \text{ atoms}$$

$$\varepsilon_F = \frac{(1.05459 \times 10^{-27})^2}{2(3 \times 1.67261 \times 10^{-24})} \times \overset{1}{(3\pi^2 \times 1.62594 \times 10^{22})^{2/3}} \times \underbrace{\frac{1}{1.6 \times 10^{-12}}}_{\text{change to eV}}$$

$$= 4 \times 10^{-4} \text{ eV} \qquad 1$$

$$T_F = \frac{\varepsilon_F}{k_B} \qquad 1$$

$$= \frac{4 \times 10^{-4} \times 1.6 \times 10^{-12}}{1.38062 \times 10^{-16}}$$

$$\simeq 5 \text{ K} \qquad 1$$

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Chapter 6

4 (a) Assume the Sun is composed entirely of Hydrogen.
i.e. number of electron = number of protons

$$N = \frac{M_s}{m_p} = \frac{2\times10^{33}\ g}{1.67\times10^{-24}\ g} = 1.2\times10^{57} \quad ① $$

$$V = \frac{4}{3}\pi R^3 = \frac{4}{3}\pi(2\times10^{9}\ cm)^3 = 3.4\times10^{28}\ cm^3 \quad ①$$

$$\mathcal{E}_F = \frac{\hbar^2}{2m}\left(3\pi^2\frac{N}{V}\right)^{2/3} \quad ①$$

$$= \frac{(1.05459\times10^{-27})^2}{2\times9.10956\times10^{-28}}\times\left(3\pi^2\times\frac{1.2\times10^{57}}{3.4\times10^{28}}\right)^{2/3}\times\frac{1}{1.6022\times10^{-12}}$$

$$\simeq 40\ keV \quad ①$$

(b) The no. of states in k-space doesn't change with relativity, thus,

$$k_F = \left(3\pi^2\frac{N}{V}\right)^{1/3} \quad ① \qquad [\text{Eqt. } (16)]$$

In the relativistic limit, $\varepsilon = pc = \hbar kc$

$$\therefore\ \mathcal{E}_F = \hbar k_F c \quad ①$$

$$= \hbar c\left(3\pi^2\frac{N}{V}\right)^{1/3}$$
$$\approx 3\hbar c\left(\frac{N}{V}\right)^{1/3}$$
$$\simeq \hbar c\left(\frac{N}{V}\right)^{1/3} \quad ①$$

(c) $V_{pulsar} = \frac{4}{3}\pi R_{pulsar}^3 = \frac{4}{3}\pi(10^6\ cm)^3 = 1.25\times10^{-10}\ V = 4.19\times10^{18}\ cm^3$

Classical: $\mathcal{E}_{F\ pulsar} = \frac{\hbar^2}{2m}\left(3\pi^2\frac{N}{V_{pulsar}}\right)^{2/3}$

$$= \frac{\hbar^2}{2m}\left(3\pi^2\frac{N}{V}\right)^{2/3}\times\left(\frac{1}{1.25\times10^{-10}}\right)^{2/3}$$

$$\simeq 160\ GeV$$

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Relativistic limit: $\mathcal{E}_{F\ pulsar} = \hbar c\left(\frac{N}{V}\right)^{1/3}$ ①

①

$$= 1.05459\times10^{-27}\times3\times10^{10}\times\left(\frac{1.2\times10^{57}}{4.19\times10^{18}}\right)^{1/3}\times\frac{1}{1.6022\times10^{-12}}$$

$$= 1.3\times10^{8}\ eV \quad ① \quad > 0.8\times10^{6}\ eV$$

which is much larger than the energy for $p + e^-$ to form $n$. Thus, hydrogen in a 'dense' star will become neutron. (calls Neutron star)