Circles

Section A

1. A circle C_1 and the circle $C_2 : x^2 + y^2 - 2x - 35 = 0$ are concentric. The area of C_1 is half of that of C_2. Find the equation of C_1.

Q1 Solution

$$C_2 : x^2 + y^2 - 2x - 35 = 0$$

i.e. $(x - 1)^2 + y^2 = 36$

$\therefore$ centre of $C_2 = (1, 0)$

radius of $C_2 = 6$

$\because$ area of $C_1 = \frac{1}{2}$ area of C_2

$\therefore$ radius of $C_1 = \frac{1}{\sqrt{2}}$ radius of C_2

$= \frac{6}{\sqrt{2}}$

$\therefore$ Equation of C_1 is

$$(x - 1)^2 + y^2 = (\frac{6}{\sqrt{2}})^2$$

i.e. $x^2 + y^2 - 2x - 17 = 0$

2. A circle, with centre G(1, –1), is tangent to the line $5x - 12y + 9 = 0$. Find the equation of the circle.

Q2 Solution

$$\text{radius} = \left|\frac{5(1) - 12(-1) + 9}{\sqrt{5^2 + 12^2}}\right|$$

$\therefore$ The required equation is

$$(x - 1)^2 + (y + 1)^2 = 2^2$$

i.e. $x^2 + y^2 + 2x + 2y - 2 = 0$

3. Find the equation of a circle C_2 which passes through the point A(4, –1) and is tangent to the circle $C_1 : x^2 + y^2 + 2x - 6y + 5 = 0$ at B(1, 2).

Q3 Solution

$$C_1 : x^2 + y^2 + 2x - 6y + 5 = 0$$

$$(x + 1)^2 + (y - 3)^2 = 5$$

$\therefore$ centre of C_1, G $= (-1, 3)$

Equation of BG is

$$y - 3 = \frac{3 - 2}{-1 - 1}(x + 1)$$

i.e. $x + 2y - 5 = 0$ . . . . (1)

Equation of the perpendicular bisector of AB is

$$(x - 4)^2 + (y + 1)^2 = (x - 1)^2 + (y - 2)^2$$

i.e. $x - y - 2 = 0$ . . . . (2)

Solving (1) and (2), $x = 3$, $y = 1$.

$\therefore$ The centre H of the required circle $= (3, 1)$.

$$AH = \sqrt{(4 - 3)^2 + (-1 - 1)^2} = \sqrt{5}$$

$\therefore$ Equation of the required circle is

$$(x - 3)^2 + (y - 1)^2 = 5$$

i.e. $x^2 + y^2 - 6x - 2y + 5 = 0$

4. A circle C passes through the intersecting points of the circles $C_1 : x^2 + y^2 = 4$ and $C_2 : x^2 + y^2 - 6x = 0$. If the point P(2, –2) is on the circle C, find the equation of C.

Q4 Solution

Let the equation of C be

$$x^2 + y^2 - 4 + k(x^2 + y^2 - 6x) = 0.$$

$\because$ P(2, −2) is on C,

$\therefore$ $2^2 + (-2)^2 - 4 + k[2^2 + (-2)^2 - 6(2)] = 0$

$\therefore$ $k = 1$

$\therefore$ The required equation is

$$x^2 + y^2 - 3x - 2 = 0.$$

5. Find the equations of tangents to the circle $x^2 + y^2 + 10x - 2y + 6 = 0$ which are parallel to the line $2x + y - 7 = 0$.

Q5 Solution

Let the required equation of tangent be

$$2x + y + C = 0\ .$$

$$C_2:\ x^2 + y^2 + 10x - 2y + 6 = 0$$

i.e. $$(x + 5)^2 + (y - 1)^2 = 20$$

$\therefore$ $$\left|\frac{2(-5)+1+C}{\sqrt{2^2+1^2}}\right| = \sqrt{20}$$

$$C - 9 = \pm 10$$

$\therefore$ $$C = -1 \text{ or } 19$$

i.e. The required equations are

$$2x + y - 1 = 0 \text{ and } 2x + y + 19 = 0.$$

6. Find the length of the chord formed by the line L: $x + 2y + 1 = 0$ on the circle C: $(x - 2)^2 + (y - 1)^2 = 25$.

Q6 Solution

C: $(x - 2)^2 + (y - 1) = 25$

$\therefore$ The coordinates of centre G = (2, 1)

radius, $r = 5$

$$\text{Distance of G from L} = \left|\frac{2+2(1)+1}{\sqrt{1^2+2^2}}\right| = \sqrt{5}$$

$$\text{The length of the chord} = 2\times\sqrt{5^2-(\sqrt{5})^2} = 4\sqrt{5}$$

7. Find the equations of the lines passing through the origin and tangent to the circle $x^2 + y^2 - 8x - 4y + 16 = 0$.

Q7 Solution

Let $y = mx$ be the required tangent.

Putting $y = mx$ into

$$x^2 + y^2 - 8x - 4y + 16 = 0,$$

$$x^2 + (mx)^2 - 8x - 4(mx) + 16 = 0$$

$\therefore$ $$(1 + m^2)x^2 - (8 + 4m)x + 16 = 0$$

As $\Delta = 0$,

$$(8 + 4m)^2 - 4(1 + m)^2 (16) = 0$$

$$3m^2 - 4m = 0$$

$\therefore$ $$m = 0 \text{ or } \frac{4}{3}$$

The required equations of tangent are

$$y = 0 \text{ and } y = \frac{4}{3}x.$$

8. **(a)** Prove that the circle $x^2 + y^2 - 2x + 6y + 1 = 0$ meets the circle $x^2 + y^2 - 8x + 8y + 31 = 0$ at two points.

(b) Find the equation of the common chord of the circles in **(a)**.

Q8 Solution

(a) $C_1 : x^2 + y^2 - 2x + 6y + 1 = 0$

centre $G_1 = (1, -3)$

radius $r_1 = \sqrt{1^2 + (-3)^2 - 1} = 3$

$C_2 : x^2 + y^2 - 8x + 8y + 31 = 0$

centre $G_2 = (4, -4)$

radius $r_2 = \sqrt{4^2 + (-4)^2 - 31} = 1$

$G_1G_2 = \sqrt{(1-4)^2 + (-3+4)^2}$

$= \sqrt{10}$

$\because$ $G_1G_2 < r_1 \; r_2$

$\therefore$ The circles C_1 and C_2 meet at two points.

(b) The equation of common chord is

$$(x^2 + y^2 - 2x + 6y + 1) - (x^2 + y^2 - 8x + 8y + 31) = 0$$
$$6x - 2y - 30 = 0$$

i.e. $3x - y - 15 = 0$

9. A variable point $P(x, y)$ moves so that the ratio of its distances from two fixed points $A(2, 0)$ and $B(0, 6)$ is a constant k.

(a) Find the equation of the locus of P.

(b) What is the locus when $k = 1$?

(c) What is the locus when $k \neq 1$?

Q9 Solution

(a)

$$PA = kPB$$
$$\sqrt{(x-2)^2 + y^2} = k\sqrt{x^2 + (y-6)^2}$$
$$x^2 - 4x + 4 + y^2 = k^2(x^2 + y^2 - 12y + 36)$$
$$(k^2 - 1)x^2 + (k^2 - 1)y^2 + 4x - 12k^2y + 12k^2 - 4 = 0$$

(b) When $k = 1$, the locus is

$4x - 12y + 36 - 4 = 0$

i.e. $x - 3y + 8 = 0$

which is a straight line.

(c) When $k \neq 1$, the locus is a circle.

10. The line $y = mx$ and the circle C: $x^2 + y^2 - 10x + 16 = 0$ intersect at two points A and B.

(a) Write down the centre and the radius of the circle C.

(b) If AB = 2, find the value of m.

Q10 Solution

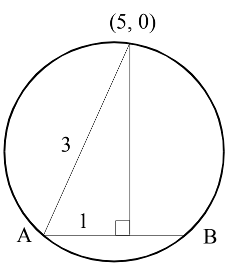
(a)

Centre of C $= (5, 0)$

Radius of C $= \sqrt{(-5)^2 - 16}$

$= 3$

(b)



The distance from (5, 0) to the line $y = mx = \left|\frac{5m}{\sqrt{1-m^2}}\right|$

$$\therefore \quad \left(\frac{5m}{\sqrt{1+m^2}}\right)^2 + \left(\frac{2}{2}\right)^2 = 3^2$$

$$25m^2 = 8 + 8m^2$$
$$m^2 = \frac{1}{2}$$
$$\therefore \quad m = \pm\frac{\sqrt{2}}{2}$$

11. A is the fixed point (5, −2) and B is a variable point on the circle C: $(x-1)^2 + (y-2)^2 = 4$. N is an internal point of division of AB such that AN : NB = 3 : 2. Find the equation of the locus of N.

Q11 Solution

Let N be (x, y) and B be (x_1, y_2).

$\because$ AN : NB = 3 : 2

$$\therefore \quad \begin{cases} x = \dfrac{2\cdot 5+3x_1}{3+2} \\ y = \dfrac{2(-2)+3y_1}{3+2} \end{cases}$$

$$\text{i.e.} \quad \begin{cases} x_1 = \dfrac{5x-10}{3} \\ y_1 = \dfrac{5y+4}{3} \end{cases}$$

$\because$ B is on the circle C,

$\therefore$ $(x_1-1)^2 + (y_1-2)^2 = 4$

$$\text{Thus} \quad \left(\frac{5x-10}{3}-1\right)^2 + \left(\frac{5y+4}{3}-2\right)^2 = 7$$

$$25x^2 + 25y^2 - 130x - 20y + 110 = 0$$

i.e. The required equation of the locus of N is

$$5x^2 + 5y^2 - 26x - 4y + 22 = 0.$$

12. Consider the line L: $(2m^2 + 6m + 1)x + (m-2)y - 3m = 0$. If L has equal intercepts on the axes, find the equation of L.

Q12 Solution

$$x\text{-intercept of L} = \frac{3m}{2m^2-6m+1}$$
$$y\text{-intercept of L} = \frac{3m}{m-2}$$
$$\therefore \quad \frac{3m}{2m^2-6m+1} = \frac{3m}{m-2}$$
$$m = 0 \quad \text{or} \quad 2m^2 + 6m + 1 = m - 2$$
$$2m^2 + 5m + 3 = 0$$
$$(2m+3)(m+1) = 0$$
$$\therefore \quad m = 0,\ -\frac{3}{2} \text{ or } -1$$

Thus, the equation of L is

$x - 2y = 0$ or $7x + 7y - 9 = 0$ or $x + y - 3 = 0$.

Section B

1. Three points P(–4, 4), Q(–1, 0), R(r, 0) are such that PQ = PR.

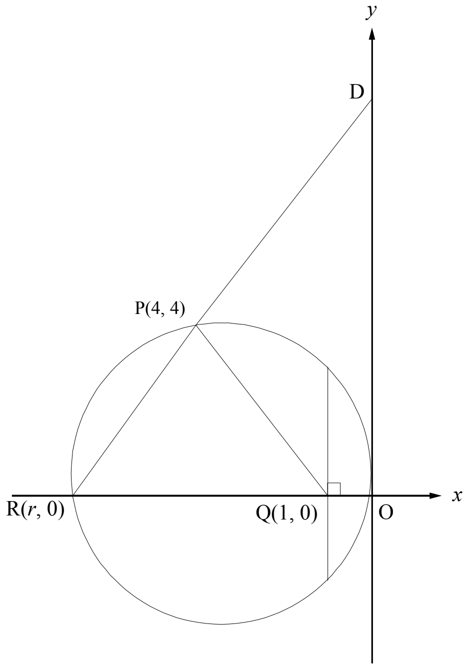
(a) Find the coordinates of the point R.

(b) Find the equation of the circle C which passes through the points P, R and the origin.

(c) A chord of the circle C is drawn through Q and perpendicular to the x-axis. Find the length of the chord.

(d) The line PR intersects the y-axis at D. Find the length of the tangent drawn from D to the circle.

Q1 Solution



(a) $\because$ PQ = PR

$(-4+1)^2 + (4-0)^2 = (-4-r)^2 + (4-0)^2$

$\therefore$ $r = -7$ or -1 (*rejected*)

$\therefore$ R is (–7, 0).

(b) Let the equation of the circle C be $x^2 + y^2 + Dx + Ey + F = 0$. Then,

$$0^2 + 0^2 + D(0) + E(0) + F = 0$$
$$(-7)^2 + 0^2 + D(-7) + E(0) + F = 0$$
$$(-4)^2 + 4^2 + D(-4) + E(4) + F = 0$$

$\therefore$ D = 7, E = –1, F = 0

$\therefore$ The equation of C is

$$x^2 + y^2 + 7x - y = 0 \quad \ldots\ldots\ldots\ldots\ (1)$$

(c) Equation of the chord is

$$x = -1. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (2)$$

Putting (2) into (1),

$$(-1)^2 + y^2 + 7(-1) - y = 0$$
$$y^2 - y - 6 = 0$$
$$(y-3)(y+2) = 0$$
$$y = 3 \text{ or } -2$$

$\therefore$ The length of the chord $= 3 - (-2)$
$= 5$

(d) Equation of PR is

$$y - 0 = \frac{4-0}{-4+7}(x+7)$$

i.e. $4x - 3y + 28 = 0$

$\therefore$ D is $(0, \frac{28}{3})$.

The length of the tangent $= \sqrt{0^2 + \left(\frac{28}{3}\right)^2 + 7(0) - \frac{28}{3}}$

$= \frac{10\sqrt{7}}{3}$

2. ΔABC is an equilateral triangle, BC is on the line $3x - 4y - 3 = 0$ and the vertex A is $(3, -1)$.

(a) Find the distance from A to BC.

(b) Find the length of a side of ΔABC.

(c) Find the coordinates of B and C.

Q2 Solution

(a) The distance d from A to BC $= \left|\frac{3(3) - 4(-1) - 3}{\sqrt{3^2 + 4^2}}\right|$

$= 2$

(b)

$$\begin{aligned} AB &= \frac{d}{\sin 60^\circ} \\ &= \frac{2}{\frac{\sqrt{3}}{2}} \\ &= \frac{4\sqrt{3}}{3} \end{aligned}$$

(c) The equation of the circle with centre A and radius AB is

$$(x - 3)^2 + (y + 1)^2 = \left(\frac{4\sqrt{3}}{3}\right)^2$$

i.e. $x^2 + y^2 - 6x + 2y + \frac{14}{3} = 0$ . . . . (1)

Solving (1) and $3x - 4y - 3 = 0$, we get

B $= \left(\frac{27 + 8\sqrt{3}}{15}, \frac{3 + 2\sqrt{3}}{5}\right)$ and C $= \left(\frac{27 - 8\sqrt{3}}{15}, \frac{3 - 2\sqrt{3}}{5}\right)$

3. Given the circle C: $x^2 + y^2 - 4x + 2y + 1 = 0$.

(a) Find the centre and radius of the circle.

(b) Show that M(3, 0) is a point inside the circle.

(c) Find the equation of the chord AB that is bisected at the point M.

(d) Write down the equation of the family of circles passing through A and B.

(e) Hence, find the equation of the circle passing through A, B and the origin.

Q3 Solution

(a) C: $x^2 + y^2 - 4x + 2y + 1 = 0$

centre G $= (2, -1)$

radius $r = \sqrt{2^2 + (-1)^2 - 1} = 2$

(b)

$$\begin{aligned} MG &= \sqrt{(3-2)^2 + (0+1)^2} \\ &= \sqrt{2} \\ &< r \end{aligned}$$

$\therefore$ M is inside the circle.

(c) $m_{MG} = \frac{0 + 1}{3 - 2} = 1$

$\because$ the required chord $\perp$ MG

$\therefore$ slope of the chord $= -1$

Equation of the chord is

$y - 0 = -1(x - 3)$

i.e. $x + y - 3 = 0$

(d) The family of circles is

$x^2 + y^2 - 4x + 2y + 1 + k(x + y - 3) = 0.$

(e) Putting (0, 0) into the equation in **(d)**,

$$1 - 3k = 0$$

$$\therefore \quad k = \frac{1}{3}$$

The required equation is

$$x^2 + y^2 + \left(\frac{1}{3} - 4\right)x + \left(\frac{1}{3} + 2\right)y = 0$$

i.e. $$3x^2 + 3y^2 - 11x + 7y = 0$$

4. Consider the circles C_1: $x^2 + y^2 + 12x - 6y + 20 = 0$
and C_2: $x^2 + y^2 - 12x + 12y - 28 = 0$.

(a) Write down the centres and radii of C_1 and C_2.
Hence, prove that C_1 and C_2 touch each other externally.

(b) **(i)** Find the equation of the common tangent to C_1 and C_2 at their point of contact.
(ii) If C_1 and C_2 touch each other at T, find the coordinates of T.

Q4 Solution

(a)
centre of C_1 = G_1 = $(-6, 3)$
radius of C_1 = r_1 = 5
centre of C_2 = G_2 = $(6, -6)$
radius of C_2 = r_2 = 10

$$\text{distance between the centres} = G_1G_2 = \sqrt{(-6-6)^2 + (3+6)^2} = 15$$

and $$r_1 + r_2 = 15$$

Thus, $$G_1G_2 = r_1 + r_2.$$

$\therefore$ C_1 and C_2 touch each other externally.

(b) **(i)** The required equation of the common tangent is

$$(x^2 + y^2 + 12x - 6y + 20) - (x^2 + y^2 - 12x + 12y - 28) = 0$$

$$24x - 18y + 48 = 0$$

i.e. $$4x - 3y + 8 = 0$$

(ii) The equation of the line G_1G_2 is

$$y - 3 = \frac{3+6}{-6-6}(x + 6)$$

i.e. $$3x + 4y + 6 = 0$$

Solving $\begin{cases} 4x - 3y + 8 = 0 \\ 3x + 4y + 6 = 0 \end{cases}$,

we have $x = -2,\ y = 0$.

$\therefore$ The coordinates of T = $(-2, 0)$

OR $\because$ GT : TG = 5 : 10 = 1 : 2

$$\therefore \quad \text{The coordinates of T} = \left(\frac{2\times(-6) + 1\times 6}{2+1},\ \frac{2\times 3 + 1\times(-6)}{2+1}\right) = (-2, 0)$$