Straight Lines

Section A

1. P is a point on the x-axis such that its distance from the origin is equal to its distance from the point (1, –2). Find the coordinates of P.

Q1 Solution

Let P be (x, 0).

Then

$$|x| = \sqrt{(x-1)^2+(0+2)^2}$$

$$x^2 = x^2 - 2x + 1 + 4$$

$$\therefore \quad x = \frac{5}{2}$$

i.e. P is $(\frac{5}{2}, 0)$.

2. Two vertices of ΔPQR are P(3, 7) and Q(–2, 5). The midpoint of PR is on the x-axis and the midpoint of QR is on the y-axis. Find the coordinates of R. Find the coordinates of the above midpoints.

Q2 Solution

Let R be (x , y).

$\because$ midpoint of QR is on the y-axis,

$$\therefore \quad \frac{-2+x}{2} = 0$$

$$\therefore \quad x = 2$$

$\because$ midpoint of PR is on the x-axis,

$$\therefore \quad \frac{7+y}{2} = 0$$

$$\therefore \quad y = -7$$

$\therefore$ R is (2, –7).

$$\text{midpoint of PR} = (\frac{3+2}{2}, \frac{7+(-7)}{2}) = (\frac{5}{2}, 0)$$

$$\text{midpoint of QR} = (\frac{-2+2}{2}, \frac{5+(-7)}{2}) = (0, -1)$$

3. The vertices of a parallelogram ABCD are A(4, y), B(–3, –1), C(–1, 3), D(x, 1). Find the values of x and y.

Q3 Solution

$\because$ midpoint of AC = midpoint of BD

$$\therefore \quad (\frac{4-1}{2}, \frac{y+3}{2}) = (\frac{-3+x}{2}, \frac{-1+1}{2})$$

$$\therefore \quad x = 6 \quad \text{and} \quad y = -3$$

4. In ΔABC, vertex A is (–1, 5), the midpoint of AB is D(1, $\frac{3}{2}$), the centroid is G(1, 2). Find the coordinates of the vertices B and C.

Q4 Solution

Let B be (x_1 , y_1) and C be (x_2 , y_2).

$\because$ AD : DB = 1 : 1

$$\therefore \quad (\frac{-1+x_1}{2}, \frac{5+y_1}{2}) = (1, \frac{3}{2})$$

$\therefore$ $x_1 = 3, \quad y_1 = -2$

$\because$ CG : GD = 2 : 1

$\therefore \quad (\frac{2(1)+x_2}{2+1}, \frac{2(\frac{3}{2})+y_2}{2+1}) = (1, 2)$

$\therefore \quad x_2 = 1, \; y_2 = 3$

i.e. B is (3, –2) and C is (1, 3).

5. The vertices of ΔABC are A(1, –1), B($1+\sqrt{3}$, $\sqrt{3}-1$), C($1-\sqrt{3}$, $\sqrt{3}+1$).

(a) Find the area of ΔABC.

(b) Find the distance from C to AB.

Q5 Solution

(a)

$$\text{Area of } \Delta ABC = \frac{1}{2}\begin{vmatrix} 1 & -1 \\ 1+\sqrt{3} & \sqrt{3}-1 \\ 1-\sqrt{3} & \sqrt{3}+1 \\ 1 & -1 \end{vmatrix}$$

$$= \frac{1}{2}\left[\sqrt{3}-1+\left(\sqrt{3}+1\right)^2+\sqrt{3}-1+\sqrt{3}+1+\left(\sqrt{3}-1\right)^2-\sqrt{3}-1\right]$$

$$= 3+\sqrt{3}$$

(b) Let h be the required distance.

$$AB = \sqrt{(1-1-\sqrt{3})^2+(-1-\sqrt{3}+1)^2}$$

$$= \sqrt{6}$$

$$\because \quad \frac{1}{2}(AB)h = \text{area of } \Delta ABC$$

$$h = \frac{2(3+\sqrt{3})}{\sqrt{6}}$$

$$= \sqrt{6}+\sqrt{2}$$

6. Two vertices of ΔABC are A(2, 3) and B(1, –1). The vertex C is on the x-axis such that ∠BAC = 45°. Find the coordinates of C.

Q6 Solution

Let C be (x, 0).

$$m_{AB} = \frac{3+1}{2-1} = 4$$

$$m_{AC} = \frac{3}{2-x}$$

$$\because \quad \angle BAC = 45°$$

$$\tan \angle BAC = \left|\frac{4-\frac{3}{2-x}}{1+4\cdot\frac{3}{2-x}}\right| = 1$$

i.e.

$$\frac{5-4x}{14-x} = \pm 1$$

$$x = -3 \text{ or } \frac{19}{5}$$

C is (–3, 0) or $\left(\frac{19}{5}, 0\right)$.

7. P(8, 3), Q(6, 7) and R(5, 1) are the midpoints of the sides BC, CA and AB of ΔABC respectively.

(a) Find the slope of PQ.

(b) Find the equation of AB.

(c) Find the distance between AB and PQ.

Q7 Solution

(a) $m_{PQ} = \dfrac{7-3}{6-8}$

$= -2$

(b) AB is the line passing R and parallel to PQ.

$\therefore$ Equation of AB is

$y - 1 = -2(x - 5)$

i.e. $2x + y - 11 = 0$

(c) Distance between AB and PQ = Distance between P and AB

$= \dfrac{2(8)+3-11}{\sqrt{2^2+1^2}}$

$= \dfrac{8\sqrt{5}}{5}$

8. Two vertices of ΔABC are A(6, 3) and B(1, –2). The vertex C(a, b) is on the line $2x - 3y + 4 = 0$.

(a) Express b in terms of a.

(b) If the area of ΔABC is 5 square units, find the coordinates of C.

Q8 Solution

(a) $\because$ C(a, b) is on the line $2x - 3y + 4 = 0$.

$\therefore$ $2a - 3b + 4 = 0$

$b = \dfrac{2a+4}{3}$

(b) $\because$ $\dfrac{1}{2}\left|\begin{vmatrix} 6 & 3 \\ 1 & -2 \\ a & b \\ 6 & 3 \end{vmatrix}\right| = 5$

$\therefore$ $-12 + b + 3a - 3 + 2a - 6b = \pm 10$

$-15 + 5a - 5 \cdot \dfrac{2a+4}{3} = \pm 10$

$\therefore$ $a = 19$ or $a = 7$

When $a = 19$, $b = \dfrac{2\times 19+4}{3} = 14$.

When $a = 7$, $b = \dfrac{2\times 7+4}{3} = 6$.

$\therefore$ The coordinates of C are (19, 14) or (7, 6).

9. The coordinates of points A and B are (–2, 2) and (1, 7) respectively. P is a point on the line segment AB such that $\dfrac{AP}{PB} = k$.

(a) Write down the coordinates of P in terms of k.

(b) Using the result of **(a)**, find the ratio that the line segment AB is divided by the line $2x - 3y + 12 = 0$.

Q9 Solution

(a) The coordinates of P are $\left(\dfrac{-2+k}{1+k}, \dfrac{2+7k}{1+k}\right)$.

(b) Let the line $2x - 3y + 12 = 0$ divides AB in the ratio $k : 1$. From **(a)**,

$2\cdot\dfrac{-2+k}{1+k} - 3\cdot\dfrac{2+7k}{1+k} + 12 = 0$

$-4 + 2k - 6 - 21k + 12 + 12k = 0$

$\therefore$ $k = \dfrac{2}{7}$

i.e. The required ratio is 2:7.

10. Lines L_1: $kx - y - k = 0$

and L$_2$: $x + ky + 1 = 0$
intersect at T, where k is a real number.

(a) Express the coordinates of T in terms of k.

(b) When k varies, find the equation of the locus of T.

Q10 Solution

(a) Solving $\begin{cases} kx - y - k = 0 \\ x + ky + 1 = 0 \end{cases}$,

we have $x = \dfrac{k^2 - 1}{k^2 + 1}$, $y = \dfrac{-2k}{k^2 + 1}$.

(b)
$$\begin{aligned} x^2 + y^2 &= \left(\frac{k^2 - 1}{k^2 + 1}\right)^2 + \left(\frac{-2k}{k^2 + 1}\right)^2 \\ &= \frac{k^4 - 2k^2 + 1}{(k^2 + 1)^2} + \frac{4k^2}{(k^2 + 1)^2} \\ &= \frac{k^2 + 2k^2 + 1}{(k^2 + 1)^2} \\ &= 1 \end{aligned}$$

The required equation of the locus of T is
$x^2 + y^2 = 1$.

11. A line L passes through the point A(2, 1) and its slope is m.

(a) Write down the equation of L.

(b) If the distance of the point (3, –2) from L is $\sqrt{2}$, find the value of m.

Q11 Solution

(a) The equation of L is
$y - 1 = m(x - 2)$
i.e. $mx - y + 1 - 2m = 0$

(b)
$$\left| \frac{m \cdot 3 - (-2) + 1 - 2m}{\sqrt{1 + m^2}} \right| = \sqrt{2}$$

$$\begin{aligned} m + 3 &= \pm\sqrt{2} \cdot \sqrt{1 + m^2} \\ \therefore \quad m^2 + 6m + 9 &= 2 + 2m^2 \\ m^2 - 6m - 7 &= 0 \\ (m - 7)(m + 1) &= 0 \\ \therefore \quad m &= 7 \text{ or } -1 \end{aligned}$$

1. In ΔABC, D is a point on BC such that AD bisects ∠BAC. The coordinates of A, B and D are (0, 4), (−3, 0) and $(\frac{1}{2}, \frac{1}{2})$ respectively.

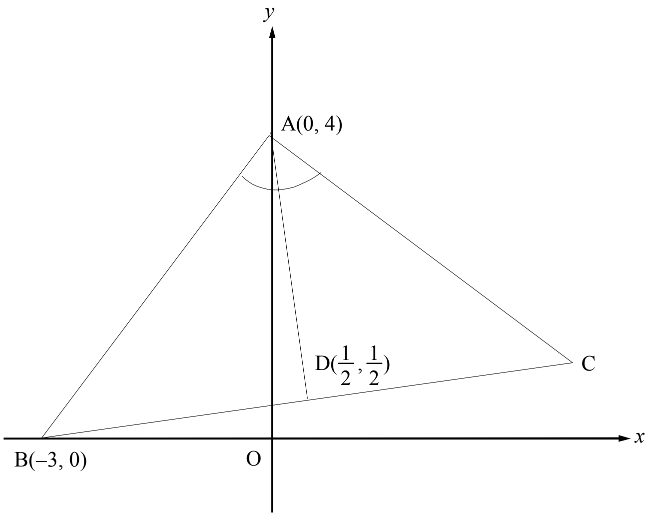
(a) Find the slopes of AB and AD.

(b) Find the slope of AC.

(c) Find the perimeter of ΔABC.

Q1 Solution

(a)



$$m_{AB} = \frac{4-0}{0+3} = \frac{4}{3}$$

$$m_{AD} = \frac{4-\frac{1}{2}}{0-\frac{1}{2}} = -7$$

(b) Let the slope of AC be m.

∵ $\angle BAD = \angle CAD$

∴ $\tan\angle BAD = \tan\angle CAD$

$$\frac{-7-\frac{4}{3}}{1+(-7)(\frac{4}{3})} = \frac{m-(-7)}{1+m(-7)}$$

∴ $m = -\frac{3}{4}$

(c) Let C be (x, y).

∵ $m_{AC} = m$

∴ $\frac{y-4}{x-0} = -\frac{3}{4}$

i.e. $3x + 4y - 16 = 0$ . . . . (1)

∵ $m_{BC} = m_{BD}$

∴ $\frac{y-0}{x+3} = \frac{\frac{1}{2}-0}{\frac{1}{2}+3}$

∴ $x - 7y + 3 = 0$ . . . . (2)

Solving (1) and (2),

$x = 4,\ y = 1.$

$\therefore$ C is (4, 1).

The perimeter of ΔABC

$= AB + BC + AC$

$= \sqrt{(0+3)^2+(4-0)^2} + \sqrt{(4+3)^2+(1-0)^2} + \sqrt{(0-4)^2+(4-1)^2}$

$= 5 + 5\sqrt{2} + 5$

$= 10 + 5\sqrt{2}$

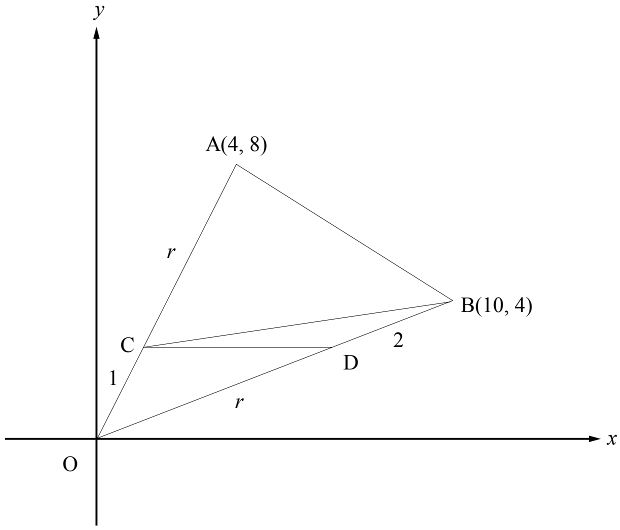
2. The coordinates of points O, A, B are (0, 0), (4, 8), (10, 4) respectively. C is a point on OA such that OC : CA = 1 : *r*. D is a point on OB such that OD : DB = *r* : 1.

(a) Express the coordinates of C and D in terms of *r*.

(b) Express the areas of ΔABC and ΔBCD in terms of *r*.

(c) If the area of ΔABC is twice of the area of ΔBCD, find the value of *r*.

Q2 Solution



(a)

$$C = \left(\frac{1(4)+r(0)}{1+r}, \frac{1(8)+r(0)}{1+r}\right) = \left(\frac{4}{1+r}, \frac{8}{1+r}\right)$$

$$D = \left(\frac{r(10)+1(0)}{1+r}, \frac{r(4)+1(0)}{1+r}\right) = \left(\frac{10r}{1+r}, \frac{4r}{1+r}\right)$$

(b)

$$\text{Area of } \Delta ABC = \frac{1}{2}\begin{vmatrix} 4 & 8 \\ \frac{4}{1+r} & \frac{8}{1+r} \\ 10 & 4 \\ 4 & 8 \end{vmatrix} = 32 - \frac{32}{1+r}$$

$$\text{Area of } \Delta BCD = \frac{1}{2}\begin{vmatrix} 10 & 4 \\ \frac{4}{1+r} & \frac{8}{1+r} \\ \frac{10r}{1+r} & \frac{4r}{1+r} \\ 10 & 4 \end{vmatrix} = \frac{32}{1+r} - \frac{32r}{(1+r)^2}$$

(c)

$$32-\frac{32}{1+r} = 2\left[\frac{32}{1+r}-\frac{32r}{(1+r)^2}\right]$$

$$\frac{2r}{(1+r)^2}-\frac{3}{1+r}+1 = 0$$

$$r^2-r-2 = 0$$

$$\therefore \quad r = 2 \text{ or } -1 \ (\textit{rejected})$$

$$\therefore \quad r = 2$$