Chapter 4

Equilibria of strategic form games

An "equilibrium" of a game in strategic form is a strategy profile that is *stable* with respect to unilateral deviations from the individual players to obtain higher payoffs. There are several methods to define stability in this respect and, therefore, there are several equilibrium concepts for strategic form games. In this chapter we essentially discuss three methods, each encompassing to a plausible notion of stability. The first is the notion of *Nash equilibrium*, which is based on the stability concept of the *best response* to the strategies chosen by the other players. Here the notion of best response refers to the assumption that each player tries to choose the best possible strategy *given* what the other players have chosen. I will discuss several different interpretations of the notion of Nash equilibrium.

The second notion of equilibrium is the concept of *equilibrium in (weakly) dominant strategies*. Here stability is based on choosing the overall best strategy, which is usually called a (weakly) dominant strategy. A dominant strategy always gives the highest payoff to a player, irrespective of what the other players do in the game.

Finally, I discuss equilibria resulting from iterated elimination of (weakly) dominated strategies, also called *iterated dominant equilibria*. The rationale behind these equilibria is that stability is based on the exclusion of certain strategies that plausibly will never be selected by the players in the game. A strategy is called "dominated" when it does worse than other strategies, irrespective of what the other players in the game do. Dominated strategies are very unlikely to be played and therefore can be eliminated by players. In the resulting "reduced" game one can repeat the procedure of elimination of dominated strategies until this is no longer possible. Nash equilibria identified in the fully reduced game are now considered to be the most plausible ones to be played. We link the method of iterated elimination of dominated strategies to the method of backward induction in the extensive form corresponding to the game under consideration.

From the stand point of the level of rationality employed in these stability concepts, Nash equilibrium is based on a rather simply, myopic kind of rationality. Equilibria in dominant strategies are based on a slightly higher level of rationality, while iterated dominant equilibria are based on a rather high level of rationality on the part of the players in the game.

In the rest of the chapter we will only consider two-person normal form games. The two players are labeled I and II. The strategy set of player I, respectively player II, is given by $S_I = \{s_1, s_2, \dots, s_n\}$, respectively $S_{II} = \{t_1, t_2, \dots, t_m\}$. So, player I has n strategies, while player II has m strategies. Each player has her own payoff function assigning to each strategy profile in the game a vNM utility value, called a "payoff." Formally, the payoff function of player I is given by $\pi_1 : S_I \times S_{II} \to \mathbb{R}$ and the payoff function of player II is given by $\pi_2 : S_I \times S_{II} \to \mathbb{R}$. Thus, for a given strategy profile $(s, t) \in S_I \times S_{II}$ the corresponding payoff vector is given by $(\pi_1(s,t), \pi_2(s,t))$. Usually these payoffs are represented in a $n \times m$-matrix as introduced in the previous chapter.

4.1 Nash equilibrium

John Nash Jr. was the first game theorist to seriously investigate a plausible equilibrium concept in normal form games based on individual rationality. Namely, when individual players try to optimize their behavior with the goal to maximize payoffs, which strategy should they choose? This depends on what one assumes about how a player perceives what the other player will do.

Nash's equilibrium concept is based on the most primitive consideration of a player: It is assumed that each player takes the strategy of the other player as given. The basic notion that expresses this behavioral rule is given by the concept of *best response* to a given strategy of the other player.

Definition 4.1 *A strategy $\widehat{s} \in S_I$ for player I is a* ***best response*** *to strategy $t \in S_{II}$ of player II if*

$$\pi_1(\widehat{s}, t) \geqq \pi_1(s, t) \qquad \text{for every } s \in S_I.$$

Similarly, a strategy $\widehat{t} \in S_{II}$ for player II is a best response to strategy $s \in S_I$ of player I if

$$\pi_1\left(s, \widehat{t}\right) \geqq \pi_1(s, t) \qquad \text{for every } t \in S_{II}.$$

Hence, a best response is a strategy that maximizes the payoff of a certain player given that the other player plays a certain strategy. To illustrate the notion of best response, consider the game represented by the following matrix:

II I	L	M	R
U	0 0	4 1	5 2
S	0 1	3 1	0 1
D	2 1	0 2	1 1

Game 4.1 - A 3×3 two-person game

In Game 4.1 each player has three strategies. The strategy sets are given by $S_I = \{U, S, D\}$ and $S_{II} = \{L, M, R\}$. The game given in Game 4.1 is structured as follows with respect to best responses:

- Strategy R is a best response for player II to player I's strategy U. This is because strategy R gives player II a payoff of 5 against strategy U of player I, while the alternative strategies only pay 0 and 4, respectively.
- Strategy M is a best response for player II to player I's strategy S.
- Strategy L is a best response for player II to player I's strategy D.
- Strategy S as well as D are best responses for player I to player II's strategy L. This shows that there might exist multiple best responses to a certain given strategy of the other player.
- Strategy D is a best response for player I to player II's strategy M.
- Strategy U is a best response for player I to player II's strategy R.

From the analysis above it might be clear that strategy U is a best response to strategy R and that strategy R is also a best response to strategy U. Hence, we have identified a "stable" strategy profile from the perspective of best response rationality. Similarly, the strategy profile (D, L) satisfies the same stability rule. Strategy profiles (U, R) and (D, L) are called the Nash equilibria of Game 4.1.

Definition 4.2 *A strategy profile* $\left(\widehat{s}, \widehat{t}\right)$ *is a **Nash equilibrium** if* $\widehat{s}$ *is a best response for player* I *to* $\widehat{t}$ *and, similarly,* $\widehat{t}$ *is a best response for player* II *to* $\widehat{s}$*. Formally, this can also be represented as*

$$\pi_1\left(\widehat{s}, \widehat{t}\right) \geqq \pi_1\left(s, \widehat{t}\right) \qquad \text{for every } s \in S_I \text{ and} \tag{4.1}$$

$$\pi_2\left(\widehat{s}, \widehat{t}\right) \geqq \pi_1\left(\widehat{s}, t\right) \qquad \text{for every } t \in S_{II}. \tag{4.2}$$

In a Nash equilibrium both players maximize their payoffs, given what the other player is doing. Hence, a Nash equilibrium is a stable state based on the myopic pursuit of self-interest.

Nash equilibrium functions usually as a "focal point" in the perception of the game by the players involved. Namely, suppose that both players play the game only once. Then the question arises how they should play the game. When the game has a unique Nash equilibrium, then this question can be answered straightforwardly: both player should play their Nash equilibrium strategies.

For example consider Game 4.2 given below. In this game there is a unique Nash equilibrium given by the strategy profile (U, R). Indeed, strategy U is a best response to strategy R, and the other way round, strategy R is a best response to strategy U. Also, strategy D

is the best response to strategy L and strategy R is the unique best response to strategy D. This implies that indeed there are no other Nash equilibria in Game 4.2.

I \ II	L	R
U	0, 0	2, 0
D	1, 1	1, 2

Game 4.2 - A 2×2 two-person game with a unique Nash equilibrium

The unique Nash equilibrium (U, R) acts as a focal point for both players for a unique way to play the game. Namely, there are no other Nash equilibria except (U, R), so both players are drawn to playing the identified strategies.

4.1.1 Games with multiple Nash equilibria

In the previous section I only considered games with a *unique* Nash equilibrium. In these games there is a clear, unambiguous way to play the game, namely by executing the uniquely determined Nash equilibrium strategies. However, there exist cases in which the Nash equilibrium is not unique. In such cases the players have no "focal point" in their analysis of the game, and will have much greater difficulty deciding about how to play the game. For example, consider Game 4.3 given below:

I \ II	L	M	R
U	0, 0	1, −1	0, 0
S	0, 0	−1, 1	−1, 1
D	1, −1	0, 0	0, 0

Game 4.3 - A 3×3 two-person game with multiple Nash equilibria.

In Game 4.3 we can identify two (2) Nash equilibria: (U, R) and (D, R). The two players in this game will have a very hard time to choose between these Nash equilibria. Namely, is anyone of these two Nash equilibria more likely to be played than the other ones? There is really no compelling reason why one should be "preferred" over the others. Thus, playing this game leads to a rather interesting dilemma.

As can be seen from the matrix representation, Game 4.3 is a zero-sum game. In this game the two Nash equilibria identified have *exactly* the same payoff vector. This is not a coincidence. In fact this is a general rule:

Proposition 4.3 (Solution Theorem) *Every Nash equilibrium strategy profile in a given constant-sum game has exactly the same payoff vector.*

The "Solution Theorem" states that every constant-sum game has a unique "value" in the sense that a unique payoff vector is identified that is supported by every Nash equilibrium in the game. In Game 4.3 this "value" is given by the payoff vector $(0, 0)$. The interpretation is that if the game is played in a plausible fashion by both players, they end up with exactly the same payoff vector, irrespective of which Nash equilibrium they exactly executed.

There are many interesting two-person games that have multiple Nash equilibria and in which there is no identified way in which the players can "obviously" play the game. A category of these type of games are given by the *coordination games*. In these games the two players are required to coordinate their strategies to obtain the highest possible payoffs.

The most famous example of a coordination game is that of *The Battle of the Sexes* game. With this game comes a story of a couple on a date. The two people that form the couple are identified as the two players in this game. The couple has to decide to go *either* to a romantic movie, identified as strategy R for both players, *or* an action movie, identified as strategy A. The man, identified as player I, prefers an action movie over a romantic movie. On the other hand, the woman, identified as player II, prefers the romantic movie over the action movie. However, if both disagree and end up in different movie theatres, both of them would not enjoy their choices at all, i.e., both players prefers each other company irrespective of which movie they go to.

The Battle of the Sexes can be represented in a matrix given in Game 4.4 below. From the matrix it is clear that coordination leads to a payoff, but coordination at different strategies leads to different payoffs for the different players.

I \ **II**	**R**	**A**
R	1, 3	0, 0
A	0, 0	3, 1

Game 4.4 - The Battle of the Sexes.

In Game 4.4 we identify both strategy profiles (R, R) and (A, A) as the two Nash equilibria. We can think of this game as that both parties have decided to meet at a particular movie theatre, but that they did not reach an agreement at which movie theatre to meet. Both players have to decide to which movie theatre to go to. Since there are multiple Nash equilibria there is indeed a true coordination problem! There is an obvious conflict of interest and there is fundamental problem of *how* to reach consensus among the players on which Nash equilibrium to coordinate.[1]

For the moment we will not try to solve the problem of how to play games with multiple equilibria. As we will discuss later in this course, a "true" solution can only be reached through the consideration and analysis of repeated play of such a game with multiple Nash equilibria.

4.1.2 Games without any Nash equilibria

In our discussion thus far we have discussed two types of games: games with a unique Nash equilibrium and games with multiple Nash equilibria. However, there is a third type of game, namely the class of games without any Nash equilibria. For example consider the zero-sum game given in Game 4.5.

I \ II	L	M	R
U	0, 0	−1, 1	1, −1
D	1, −1	1, −1	0, 0

Game 4.5 - A 2×3 two-person game without Nash equilibria.

In Game 4.5 there are no Nash equilibria.[2] Hence, we are not able to identify strategy profiles that are stable in the sense that players play a best response to the other player's strategy. Nevertheless players are expected to make certain choices in this game when they are set to play it. Since there are no Nash equilibria, we might conclude that there is no stable state in this game.

Instead the players are doomed to play in "cycles" when ever they follow best response strategies. For example, in Game 4.5 for player I strategy U is a best response to strategy R and strategy D is a best response to strategy M. For player II, however, strategy M is a best response to strategy U and strategy R is a best response to strategy D. If both players use their best responses to whatever the other player plays, then we get a cyclic reasoning in the game. This cycle is given by the sequence

$$(U, M) \to (D, M) \to (D, R) \to (U, R) \to (U, M) \to \cdots \tag{4.3}$$

[1] Even if both players are able to communicate with each other, the fundamental coordination problem persists. The main question is now: Why? Well, the reason is that it is assumed in noncooperative game theory that there are *no* binding agreements possible in a game. Hence, coordination can *not* be enforced through an agreement among the players. This is explored further in the problem section of this chapter.

[2] Note that the "Solution Theorem" applies to this game as well, even though it does not have any Nash equilibria. Namely, since there are no strategy profiles that can be identified as Nash equilibria in this game, any statement about these strategy profiles is therefore true.

Thus, both players are doomed to chase each other's tail until they drop...

In the next chapter we will see that we still can identify a certain (modified) type of Nash equilibrium for a game without any obvious Nash equilibria.[3] Therefore, I postpone discussion of this type of games till the next chapter.

4.1.3 Strict Nash equilibria

Finally, I consider a special subclass of Nash equilibria. Consider the two-person game given as Game 4.6 below.

II / I	**L**	**R**
U	0 0	0 2
D	1 0	2 3

Game 4.6 - Strict Nash equilibrium

In Game 4.6 there are two Nash equilibria, (U, L) and (D, R). From the matrix representation of this game it is clear that these two Nash equilibria have a very different nature. Namely, (U, L) is only a Nash equilibrium because by deviating to strategy D, respectively strategy R, player I, respectively player II, can *not strictly* improve their payoffs. Indeed, if any player deviates she only gets a payoff that is equal to the equilibrium payoff, thus not "beating" the identified Nash equilibrium strategy.

It is a different story for Nash equilibrium (D, R). In this Nash equilibrium a deviation from strategy D to strategy U leads to a *strictly* lower payoff for player I. Similarly, for player II a deviation from strategy R to strategy L worsens the payoff strictly. In this respect Nash equilibrium (D, R) is much more compelling than Nash equilibrium (U, L).

Formally we call the strategy profile (D, R) a *strict Nash equilibrium* to distinguish it from a Nash equilibrium with weaker improvement properties, such as strategy profile (U, L). We modify the formal definition of a Nash equilibrium to capture this idea as follows:

Definition 4.4 *A strategy profile* $\left(\widehat{s}, \widehat{t}\right)$ *is a* ***strict Nash equilibrium*** *if* $\widehat{s}$ *is a strict best response for player* I *to* $\widehat{t}$ *and, similarly,* $\widehat{t}$ *is a strict best response for player* II *to* $\widehat{s}$*, in the sense that*

$$\pi_1\left(\widehat{s}, \widehat{t}\right) > \pi_1\left(s, \widehat{t}\right) \quad \text{for every } s \neq \widehat{s} \text{ and} \tag{4.4}$$

$$\pi_2\left(\widehat{s}, \widehat{t}\right) > \pi_1\left(\widehat{s}, t\right) \quad \text{for every } t \neq \widehat{t}. \tag{4.5}$$

[3] In fact the "solution" to this dilemma is to give a certain description to such cycles as identified for Game 4.5.

We will not very frequently use the notion of a strict Nash equilibrium in our discussion in the next chapters of these lecture notes. However, once in a while I will call upon this notion to distinguish certain identified strategy profiles from a regular Nash equilibrium. I already will do this in the next section.

4.2 Strategic domination

The Nash equilibrium concept is based on the rationality concept of "Best Response." Namely, in a Nash equilibrium each player plays a best response to the other's strategy. This type of rationality is based on a rather limited perception of the game on the part of the players. Why do they take the strategy of the other as given? In principle there is no compelling reason why this should be the case, and in that respect, the applied behavioral standard.

For example, we can consider a very simple two-person game in which this higher order of rationality is applicable. Consider a game in which each player can either "cooperate" with the other player or "defect." If both players cooperate, they both reap large gains. However, if one player cooperates and the other player defects, the defector obtains even larger gains while the cooperator suffers significant losses. Finally, if both players defect they only make very modest gains. This game is called the *Prisoners' Dilemma* and has many applications in economics, biology, and political science. A formal representation of the Prisoners' Dilemma is given in Game 4.7:

II **I**	**C**	**D**
C	10 10	20 0
D	0 20	1 1

Game 4.7 - The Prisoners' Dilemma

The Prisoners' Dilemma is a very interesting game. On the one hand, the optimal outcome is obvious. Both players should cooperate (strategy C) and reap large gains from their mutual cooperation. Namely, it is also obvious that if one of the players defects, the other retaliates immediately by defecting as well, forcing both of them to much lower profit levels.

On the other hand, however, the game has only one Nash equilibrium, which is strict. Namely, (D, D) is the unique Nash equilibrium of this game. Viewed from an individualistic point of view, the game is very simple as well. For both players strategy D is strictly better than strategy C: Whatever the other player does, it is always better to select strategy D over strategy C, since it guarantees a strictly higher payoff. Thus, strategy D strictly "dominates" strategy C for both players.

From the previous paragraphs the "dilemma" for both players becomes clear. Do the players go for the optimal strategy profile (C, C), or do they go for the myopically rational and compelling strict Nash equilibrium (D, D)? It might be clear that if these players play the game only once, the Nash equilibrium seems the logical way to go. However, if the game is repeated many times, both players might try to establish the optimum (C, C) as a regular outcome in their play.

In the Prisoners' Dilemma I consider a more advanced type of rationality, namely one in which the players try to select a strategy that is the "best" irrespective of what the other player has chosen. This form of reasoning is captured in the next definition.

Definition 4.5 *For player I a strategy $\widehat{s} \in S_I$ is said to* ***strongly dominate*** *strategy $s \in S_I$ if for every strategy $t \in S_{II}$ for player II it holds that*

$$\pi_1(\widehat{s}, t) > \pi_1(s, t).$$

Similarly, for player II a strategy $\widehat{t} \in S_{II}$ is said to strongly dominate strategy $t \in S_{II}$ if for every strategy $s \in S_I$ for player I it holds that

$$\pi_2\left(s, \widehat{t}\right) > \pi_2(s, t).$$

A strategy $\widehat{s} \in S_I$ is said to be ***(strongly) dominant for player*** **I** *if it dominates every other strategy $s \in S_I$.*
A strategy $\widehat{t} \in S_{II}$ is said to be ***(strongly) dominant for player*** **II** *if it dominates every other strategy $t \in S_{II}$.*[4]

For player I we say that a strategy $s \in S_I$ is **dominated** if there is some other strategy $\widehat{s} \in S_I$ that strongly dominates it. Similarly, we define dominated strategies for player II.

From our analysis of the Prisoners' Dilemma as given in Game 4.7, we concluded that for both players strategy D is strongly dominant. Hence, (D, D) is a Nash equilibrium "in dominant strategies." We also identified (D, D) as the unique Nash equilibrium in the Prisoners' Dilemma. this is no coincidence. Namely, the following general statement follows immediately from some casual observation of the notions presented above.

Proposition 4.6 *In any game in strategic form the following statements are true.*

1. *Each player has at most one (strongly) dominant strategy.*
2. *If each player has a dominant strategy, then the resulting strategy profile is called an* ***equilibrium in dominant strategies****. Hence, in any game there is at most one equilibrium in dominant strategies.*

[4] Throughout these lecture notes I will use the term "dominant" interchangeably with the term "strongly dominant."

3. *If there exists an equilibrium in dominant strategies, then the equilibrium in dominant strategies is a strict Nash equilibrium, which is equal to the game's unique Nash equilibrium.*

Next we turn to a slightly weaker form of domination as defined and discussed above. Again I consider a Prisoners' Dilemma, but in this case I have modified the payoffs slightly to reduce the Nash equilibrium (D, D) from being a *strict* Nash equilibrium to being a regular equilibrium. This is represented in the following matrix:

I \ **II**	**C**	**D**
C	10 10	20 1
D	1 20	1 1

Game 4.8 - Modified Prisoners' Dilemma

In Game 4.8 there is again a unique Nash equilibrium, namely the strategy profile (D, D). However, since payoffs are modified to reflect an equal payoff for strategies D and C if the other play selects strategy D. In other words, in Game 4.8 for either player strategies C as well as D are best responses to strategy D.

It is obvious that strategy D is no longer strongly dominant, but it still has similar features as before. It is still the most plausible strategy to choose, since it guarantees a player payoffs that are "maximal" irrespective of what the other player selects.[5] This is captured in the notion of *weak domination*.

Definition 4.7 *For player I a strategy $\widehat{s} \in S_I$ is said to* ***weakly dominate*** *strategy $s \in S_I$ if for every strategy $t \in S_{II}$ for player II it holds that*

$$\pi_2(\widehat{s}, t) \geqq \pi_2(s, t),$$

and for some strategy $t' \in S_{II}$ for player II it holds that

$$\pi_2(\widehat{s}, t') > \pi_2(s, t').$$

Similarly, for player II a strategy $\widehat{t} \in S_{II}$ is said to weakly dominate strategy $t \in S_{II}$ if for every strategy $s \in S_I$ for player I it holds that

$$\pi_2(s, \widehat{t}) \geqq \pi_2(s, t)$$

[5] I define a outcome to be "best" if its payoff to a certain player is *strictly larger* than any of the alternative outcomes. Similarly, I define an outcome to be "maximal" if its payoff is *at least as high as* the payoff guaranteed by any other outcome.

and for some strategy $s' \in S_I$ for player I it holds that

$$\pi_2\left(s', \widehat{t}\right) > \pi_2\left(s', t\right).$$

A strategy $\widehat{s} \in S_I$ is said to be ***weakly dominant for player* I** *if it weakly dominates every other strategy $s \in S_I$.*
A strategy $\widehat{t} \in S_{II}$ is said to be ***weakly dominant for player* II** *if it weakly dominates every other strategy $t \in S_{II}$.*

For player I we say that a strategy $s \in S_I$ is **weakly dominated** if there is some other strategy $\widehat{s} \in S_I$ that weakly dominates it. Similarly, we define weakly dominated strategies for player II.

The main difference between weak domination and (strong) domination is that by focussing on a weakly dominant strategy only, certain Nash equilibria might be deselected. For example consider the two-person game given in Game 4.9:

I \ II	L	M	R
U	10, 10	1, 10	0, 0
S	10, 1	2, 2	0, 2
D	0, 0	0, 1	0, 0

Game 4.9 - Nash equilibrium in weakly dominant strategies

Game 4.9 is an extension and enhancement of the modified Prisoners' Dilemma given in Game 4.8. In Game 4.9 we identify two Nash equilibria, namely (U, L) and (S, M). Both are regular Nash equilibria. However, the Nash equilibrium (S, M) is an *equilibrium in weakly dominant strategies*. Indeed, for player I strategy S is weakly dominating the alternative strategies U and D. Similarly, for player II strategy M is weakly dominating alternative strategies L and R. The Nash equilibrium (U, L) is also called a *Nash equilibrium in dominated strategies* to distinguish it from the other Nash equilibria in the game. I emphasize that if a Nash equilibrium is composed of weakly dominated strategies it is obviously inferior in the sense that these (dominated) strategies are less likely to be played by the players.

Note that we have here in Game 4.9 a situation in which there are multiple Nash equilibria of which one has specific properties, in the sense that the Nash equilibrium is identified by the weakly dominant strategies for both players.

Proposition 4.8 *In any game in strategic form, each player has at most one weakly dominant strategy.*
If each player has a weakly dominant strategy, then the resulting strategy profile is called an ***equilibrium in weakly dominant strategies****. Hence, in any game there is at most one equilibrium in weakly dominant strategies.*

The example given in Game 4.9 makes clear that if there exists an equilibrium in weakly dominant strategies it does not have to coincide with the unique Nash equilibrium. It does not even have to be a strict Nash equilibrium. Namely, the equilibrium in weakly dominant strategies (S, M) in Game 4.9 is not strict, since both strategy U and S is also a best response to strategy M and strategies L, M, and R are all best responses to strategy S. This implies that Proposition 4.8 cannot be strengthened into similar conclusions we made for equilibria in (strongly) dominant strategies reported in Proposition 4.6.

4.3 Iterated elimination of dominated strategies

The rationality that we have applied thus far to identifying an appropriate way to play a game is by looking for the "best" strategies. In particular we discussed what it meant for a certain strategy to be the "best" one. In Nash equilibrium we identified the *best responses* to a given strategy that could be played by the other played. The argument here is that a player should always optimize payoffs given what the other player does. Next we considered *(weakly) dominant* strategies as those that are best responses to any strategy that the other player could choose.

The rationality concept to be developed in this section is one based on a reversed notion, namely the exclusion of "bad" strategies from selection by a player. The rational here is that a player should never play a "bad" strategy. By eliminating "bad" strategies for consideration by a player, we identify a way in which players will play the game.

Here a "bad" strategy can be defined in two ways: it could be a (strongly) dominated strategy, or it could be a weakly dominated strategy. Each of these possible definitions will be explored in the next subsections.

4.3.1 Iterated strong dominance

If in a strategic game, a certain strategy is strongly dominated by another strategy, then by definition this means that the dominated strategy is always worse than that alternative strategy *irrespective of what strategy the other player selects*. Hence, a dominated strategy from the viewpoint of a player generates always strictly lower payoffs than an alternative strategy. So, in a strategic game that is played only once, indeed it seems logical to say that a rational player should never play a dominated strategy.[6] This reasoning forms the cornerstone of the following method to "solve" a static game in strategic form.

[6] As we have seen in the discussion of the prisoners' dilemma game (PD), this is certainly not always a plausible way to reason and proceed. In the PD there exists a social optimum that gives strictly higher payoffs than the equilibrium in dominant strategies. Nevertheless if the game is only played once, then it is plausible to reason that a rational player should never play a dominated strategy. This becomes different when we consider a game to be played repeatedly. In the repeated PD playing the dominated strategy becomes a very reasonable option as we will discuss in the chapter on repeated games.

If we exclude that a player selects any of his strongly dominated strategies, then we in fact reduce his strategy set. Namely, the reduced strategy set consists of those strategies that are not strongly dominated for that particular player. If we consider these reduced strategy sets for both players in the game, we have created a "reduced" game. Within this reduced game we could again consider the set of strategies that are not dominated by an alternative strategy in the reduced game. Thus, we get a repeated procedure of reducing the strategy set for each player in an iterated fashion. This procedure stops when we obtain a game that cannot be reduced any further by deleting dominated strategies. To define this elimination procedure I introduce an additional concept.

Definition 4.9 *A strategy $\widehat{s} \in S_1$ for player I is **undominated** when there does not exist an alternative strategy in S_1 that strongly dominates strategy $\widehat{s}$.*
*Similarly, a strategy $\widehat{t} \in S_2$ for player II is **undominated** when there does not exist an alternative strategy in S_2 that strongly dominates strategy $\widehat{t}$.*

Consider a two-person game in strategic form. The method of iterated elimination of dominated strategies is now describes as follows:

1. We define $S_1^{(1)} \equiv S_1$ and $S_2^{(1)} \equiv S_2$. The reduced game of stage 1 is not given by the strategy sets $S_1^{(1)}$ and $S_2^{(1)}$, and the payoff functions for both players.

2. Let the reduced game of stage k be given. Then we define

$$\begin{aligned} S_1^{(k+1)} &= \left\{ s \in S_1^{(k)} \mid s \text{ is undominated in the reduced game for player } I \right\} \\ S_2^{(k+1)} &= \left\{ s \in S_2^{(k)} \mid s \text{ is undominated in the reduced game for player } II \right\} \end{aligned}$$

3. Define the reduced game of stage $(k+1)$ by the reduced strategy sets $S_1^{(k+1)}$ and $S_2^{(k+1)}$, and the payoff functions restricted to the reduced set of strategy profiles given by $S_1^{(k+1)} \times S_2^{(k+1)}$.

4. If $S_1^{(k+1)} = S_1^{(k)}$ *as well as* $S_2^{(k+1)} = S_2^{(k)}$ stop the procedure. Otherwise return to step 2 of the procedure.

The game that is created by application of the method of iterated elimination of strongly dominated strategies is called the *weakly irreducible game* corresponding to the original strategic form game.

To make this method of iterated elimination clear we apply it to a two-person game.

Consider the game given by the following matrix:

II I	**N**	**E**	**S**	**W**
U	30 25	25 20	10 80	40 25
M	60 35	45 25	20 20	50 30
D	95 30	90 20	30 95	60 20

(4.6)

Game 4.10

Note that this game does not have any equilibria in dominant or weakly dominant strategies. However, it has a unique Nash equilibrium given by (M, N).

Now I will subject this game to the method of iterated elimination of (strongly) dominated strategies. In the following I will describe each step in this process in detail. All reduced games are represented by their matrix form.

Step 1 The reduced game of stage 1 is exactly equal to the matrix as given in matrix (4.6). Hence, in this first step the reduced game is exactly equal to the original game.

Step 2 In the reduced game of stage 1 we can see that player I does not have any strongly dominated strategies. On the other hand, player II has a strongly dominated strategy, namely strategy S. Thus the reduced strategy sets are given by $S_1^{(2)} = \{U, M, D\}$ and $S_2^{(2)} = \{N, E, W\}$.

Step 3 The reduced game of stage 2 is now given by the following matrix:

II I	**N**	**E**	**W**
U	30 25	25 20	40 25
M	60 35	45 25	50 30
D	95 30	90 20	60 20

(4.7)

This matrix is found by eliminating the column given by the dominated strategy S.

Step 4 The reduced game of stage 2 given by matrix (4.7) can be analyzed again. We immediately see that for player I strategies U and D are both strongly dominated by strategy M. Also we note that player II does not have any strongly dominated strategies. Hence, the reduced strategy sets are given by $S_1^{(3)} = \{M\}$ and $S_2^{(3)} = \{N, E, W\}$.

Step 5 The reduced game of stage 3 is in turn given by eliminating the first and third rows in matrix (4.7), corresponding to strategy U, respectively D for player I. This leads to the following matrix representation of the reduced game of stage 3:

I \ II	N	E	W
M	60 35	45 25	50 30

(4.8)

Step 6 The reduced game of stage 3 given by matrix (4.8) can be subjected to the same analysis. We easily observe that for player II strategies E and W are both strongly dominated by strategy N. Hence, the reduced strategy sets are given by $S_1^{(4)} = \{M\}$ and $S_2^{(4)} = \{N\}$.

Step 7 Finally, the reduced game of stage 4 is derived by eliminating the second and third columns in matrix (4.8). This leads to the following representation:

I \ II	N
M	60 35

(4.9)

The game given above is clearly weakly irreducible and identifies exactly the unique Nash equilibrium of the original game.

The method sketched above and applied to Game 4.10 given by matrix (4.6) identifies a unique equilibrium in this case. This unique equilibrium is called the *Iterated Dominant Equilibrium*. In other cases, however, it is very well possible that the method of iterated elimination of strongly dominated strategies identifies a unique equilibrium.

For example, consider the game given by

I \ II	N	E	S	W
U	30 25	35 30	10 80	40 25
M	60 35	45 25	20 20	50 30
D	95 30	90 20	30 95	60 20

(4.10)

Game 4.11

Game 4.11 only differs from Game 4.10 considered before in the fact that strategy profile (U, E) has a different payoff. After application of the method of iterated elimination of strongly dominated strategies we arrive at a weakly irreducible game given by the matrix

II I	**N**	**E**
U	30 25	35 30
M	60 35	45 25

(4.11)

Here the application of the method of iterated elimination of strongly dominated strategies did not lead to the identification of a unique iterated dominant equilibrium. Instead we have derived an irreducible 2×2-game with two Nash equilibria, namely (M, N) and (U, E). Note that these Nash equilibria are also exactly the Nash equilibria of the original game given by matrix (4.11). This is no coincidence, of course, since eliminating strongly dominated strategies does not affect the Nash equilibria of the game. This leads to the statement of the following properties:

Proposition 4.10 *Consider any game in strategic form.*

1. *If application of the method of iterated elimination of (strongly) dominated strategies identifies a unique iterated dominant equilibrium, then this equilibrium is equal to the unique Nash equilibrium of the original game.*
2. *If application of the method of iterated elimination of (strongly) dominated strategies leads to a weakly irreducible game with multiple Nash equilibria, then the Nash equilibria of the weakly irreducible game are exactly equal to the Nash equilibria of the original game.*

Application of the method of iterated elimination of dominated strategies is a way to reduce the size of the decision problems facing the players. At worst we arrive at an irreducible game that is easier to understand than the original game and in which the Nash equilibria are exactly the ones of the original game. On the other hand, application of this method in certain cases might lead to an iterated dominant equilibrium, which completely "solves" the game.

Consider the following game:

II I	**N**	**E**	**S**	**W**
U	40 30	40 30	30 80	30 25
M	40 30	40 30	30 20	35 30
D	95 25	90 20	90 95	95 20

(4.12)

Game 4.12

Game 4.12 is a modification of Game 4.11. After application of the method of iterated elimination of strongly dominated strategies we arrive at a weakly irreducible game given by the matrix

I \ II	N	E
U	30, 40	30, 40
M	30, 40	30, 40

(4.13)

In this particular case the weakly irreducible game consists of Nash equilibria only, in this exactly the four Nash equilibria of the original game. Additionally all of these Nash equilibria have exactly the same payoff for the two players.[7]

Although the resulting irreducible game does not constitute a 1×1-matrix, it is clear that we can argue that we solved the game nevertheless. In case of Game 4.12 we identified four Nash equilibria with exactly the same payoffs and with interchangeable strategies, i.e., if (s_1, t_1) as well as (s_2, t_2) are Nash equilibria, then (s_1, t_2) and (s_2, t_1) are Nash equilibria as well. This implies that players can select any Nash equilibrium strategy to arrive at a Nash equilibrium with exactly the same payoff.

This is summarized in the following definition:

Definition 4.11 *A game is called* ***strongly dominance solvable*** *if it can be reduced through iterated elimination of strongly dominated strategies to a weakly irreducible game consisting of Nash equilibria only.*

It is clear from the definition that a strongly dominance solvable game can be "solved" completely by iterated elimination of strongly dominated strategies.

4.3.2 Iterated weak dominance

In the previous subsection I considered the method of iterated elimination of dominated strategies. The rationale behind this method was straightforward: if a strategy is dominated, it is reasonable to expect that a rational player would never select it. Thus leading to the reduction of the game to the weakly irreducible game. Clearly this method is "tight" in the sense that there is a unique weakly irreducible game associated with every strategic form game.

In this subsection I consider the method of iterated elimination of weakly dominated strategies. The idea behind this method is, however, less straightforward. Namely, the reasoning is now that when a strategy is weakly dominated a rational player should not

[7] This is not surprising, since if a game consists of Nash equilibria only, the payoffs in these equilibria have to be *exactly* the same.

select this strategy. As we have seen in our discussion of weak domination, there exist Nash equilibria in weakly dominated strategies that seem perfectly reasonable at first sight. Hence, when we apply the method of iterated elimination of weakly dominated strategies some caution should be applied as well.

Also the rationale seems to be less compelling than in the case of iterated elimination of strongly dominated strategies. If a strategy is weakly dominated it is certainly not clear that under *any* circumstances a player would not be willing to choose it. It might well be that it makes sense to keep one's options open and include weakly dominated strategies in the set of strategies that one wishes to consider for play in a game.

As we will see, this discussion is boiled down into a rule that the **order** in which weakly dominated strategies are eliminated is very important and might lead to different irreducible games! I will first illustrate this point with the use of two examples.

Consider first Game 4.13 as given by the matrix below.

I \ **II**	**X**	**Y**	**Z**	**T**
N	2 2	0 4	0 4	0 4
W	4 0	3 3	1 5	1 5
S	4 0	5 1	4 4	2 6
E	4 0	5 1	5 2	5 6

(4.14)

Game 4.13

In Game 4.13 we can identify two Nash equilibria, (A, X) and (D, T). Obviously (A, X) is a strict Nash equilibrium, while (D, T) is a Nash equilibrium in weakly dominated strategies. Indeed, strategy D is weakly dominated by strategy C, while strategy T is weakly dominated by strategy Z.

When we apply the method of iterated elimination of weakly dominated strategies, it can be done only in one, unique fashion. First, the weakly dominated strategies D and T are eliminated. In the reduced game we then remove strategies C and Z. Finally, we remove the weakly dominated strategies B and Y. Thus, the application of the method has identified in a unique fashion the Nash equilibrium (A, X).

The Nash equilibrium (A, X) is called an iterated weakly dominant equilibrium. In Game 4.13 this equilibrium is indeed unique, but as the next example shows this does have to be the case at all. The order of elimination of weakly dominated strategies is of crucial importance in the identification of such equilibria.

Consider the strategic game given by the following matrix:

I \ II	N	E	S	W
U	3 1	4 0	2 3	2 1
D	2 2	2 0	1 3	2 2

(4.15)

Game 4.14

Game 4.14 has four (4) Nash equilibria: (U, E), (D, N), (D, E), and (D, W). Of these four Nash equilibria, only (D, E) is an equilibrium in weakly dominant strategies. The other three Nash equilibria seem to be "less relevant."

What can we say with the application of the method of iterated elimination of weakly dominated strategies? First, we can identify three orders in which we can execute this method. Each of these different orders identifies different sets of equilibria. I discuss these three different orders one after the other:

Order A For player I we identify strategy D as a weakly dominant strategy. We thus eliminate strategy U. In the reduced game we subsequently remove strategy S for player II since it is dominated by the other strategies. Thus we have arrived at an irreducible game consisting of three Nash equilibria of the original game: (D, N), (D, E), and (D, W).

Order B For player II we identify strategy E as a weakly dominant strategy. After eliminating strategies N, S, and W we have arrived at another irreducible game composed of two Nash equilibria of the original game, namely (D, N) and (D, E).

Order C Finally, we identify for each player a weakly dominant strategy, namely strategy D for player I and strategy E for player II. Simultaneously we are thus allowed to remove all other strategies since these are weakly dominated. This gives us a third irreducible game consisting of the equilibrium in weakly dominant strategies (D, E) only.

Although Game 4.14 is rather straightforward in structure, I identified three explicitly different orders in which we can remove weakly dominated strategies, each order identifying another irreducible game and different sets of Nash equilibria of the original game. The common denominator of these three different orders is that the equilibrium in weakly dominant strategies (D, E) is selected by all three of the orders of application. After some deliberation it becomes clear that this is not be a surprise. Namely, if one eliminates weakly dominated strategies, one will never eliminate a weakly dominant strategy.

Finally, consider another example of a strategic game, this one without any weakly dominant strategies. In Game 4.15 we identify four Nash equilibria: (S, M), (U, M), (S, R), and

(D, R).

I \ II	L	M	R
U	0, 2	1, 3	1, 2
S	0, 2	1, 4	2, 4
D	1, 1	0, 1	2, 2

(4.16)

Game 4.15

There are essentially three orders in which we can eliminate weakly dominated strategies.

Order 1 Starting with player I, we note that strategy S weakly dominates strategy U. In the reduced game, after eliminating strategy U, strategy R is a weakly dominant strategy for player II. The irreducible game derived in this fashion identifies the Nash equilibria (S, R) and (D, R).

Order 2 Starting with player II, we find that strategy M weakly dominates strategy L. After eliminating L, in the reduced game we see that strategy S is weakly dominant for player I. The resulting irreducible game now identifies Nash equilibria (S, M) and (S, R).

Order 3 We can also start by considering the two players simultaneously. Here, both strategies U and L are eliminated simultaneously. In the reduced game we see that there is an equilibrium in weakly dominant strategies, (S, R).

There are two remarks to be made with Game 4.15. First, all different orders in this example identify equilibrium (S, R), although at first sight this Nash equilibrium does not have any specific features that makes it more special than any of the other Nash equilibria in the original game.[8] Second, the identified Nash equilibrium (U, M) is not identified by any of the three orders of elimination investigated. The reason for this is that strategy U is a weakly dominated strategy for player I, thus rendering this equilibrium less appealing than the other Nash equilibria of Game 4.15.

From the discussion in this subsection it is clear that the method of iterated elimination of *weakly* dominated strategies is much more sensitive than the method of iterated elimination of *strongly* dominated strategies. When one decides to apply the method of iterated elimination of weakly dominated strategies it is very much advised to investigate all orders of elimination carefully. Different orders might identify strictly different sets of Nash equilibria.

[8] From the matrix it is clear that (S, R) is neither a strict Nash equilibrium nor an equilibrium in weakly dominant strategies.

4.3.3 Dominance solvability

In the previous pages we discussed in general the method of iterated elimination of weakly dominated strategies. Our main conclusion is that the order in which we eliminate strategies is very important. Different orders of elimination usually lead to the identification of different sets of Nash equilibria.

In this subsection we focus on a particular order of elimination, namely the maximal elimination of weakly dominated strategies. In this elimination process at every stage all weakly dominated strategies of each players are eliminated. This forms the foundation of the following definition.

Definition 4.12 *Consider any matrix form game* Γ*.*

(a) The ***strongly irreducible subgame*** *of* Γ *is the (sub) matrix of all outcomes in the game* Γ *that survive the iterative procedure in which* all *the weakly dominated strategies of* both *players are eliminated at each stage of the elimination procedure.*

(b) The game Γ *is (weakly)* ***dominance solvable*** *if all players are indifferent between all outcomes in its strongly irreducible subgame. Thus, in a dominance solvable game the surviving outcomes after maximal elimination of weakly dominated strategies are Nash equilibria of the original game that generate exactly the same payoffs.*

Next we consider an example that illustrates the definition of dominance solvability of a game. Let us look at the following game:

I \ II	N	E	S	W
U	5, 5	5, 5	8, 2	3, 5
M	2, 4	2, 3	2, 3	4, 4
D	4, 1	2, 4	8, 4	2, 0

(4.17)

Game 4.16

From the matrix it is clear that Game 4.16 has four Nash equilibria: (U, N), (U, E), (D, S), and (M, W).

The maximal elimination procedure of weakly dominated strategies is performed in two steps. In the first step, we observe that for player I U weakly dominates D and that for player II N weakly dominates W and E weakly dominates S. After elimination of D, W, and S we observe that in the reduced game strategy U strongly dominates M for player I. Thus, in the second step strategy M is eliminated. This results in the strongly irreducible game given by

II **I**	**N**	**E**
U	5 5	5 5

(4.18)

This implies that Game 4.16 is indeed dominance solvable and that the resulting payoff is $(5, 5)$. This payoff profile is supported by two Nash equilibria, (U, N) and (U, E).

As a special case of dominance solvability I remark that any game that has a Nash equilibrium in weakly dominant strategies is dominance solvable. Furthermore, the identified payoff is the one corresponding to the equilibrium in weakly dominant strategies.

4.3.4 Domination and backward induction

In this section we explore the remarkable similarity of the method of backward induction in extensive form games and the method of iterated elimination of (weakly) dominated strategies for strategic form games. Namely, both methods turn out to eliminate inferior strategies, but in the different representations of the game.

We start off by considering a plausible example of a game that we can study in its extensive form as well as its normal form. The extensive form of our example is given by the game tree depicted in Figure 4.1. Obviously we may apply the method of backward induction, since it is a game with complete information. The identified paths in the game tree are highlighted in Figure 4.1.

Next we construct the strategic form representation of the game given in Figure 4.1. Obviously player I has two strategies and his strategy set is thus given by $S_1 = \{U, D\}$. On the other hand, player II has two decision moments, each with two choices. This implies that she has four strategies and her strategy set is given by $S_2 = \{WE, WS, NE, NS\}$. We are now able to construct a 2×4-matrix representation of the strategic form of this game.

II **I**	**WE**	**WS**	**NE**	**NS**
U	2 3	2 3	1 0	1 0
D	3 2	0 1	3 2	0 1

(4.19)

We identify three Nash equilibria in matrix (4.19), namely (U, WE), (U, WS), and (D, NE). We also identify strategy WE as a weakly dominant strategy for player II. Namely, strategy WE weakly dominates strategies WS and NE and strongly dominates strategy NS. However, player I does not have a weakly dominant strategy. This implies that the game does not have an equilibrium in (weakly) dominant strategies.

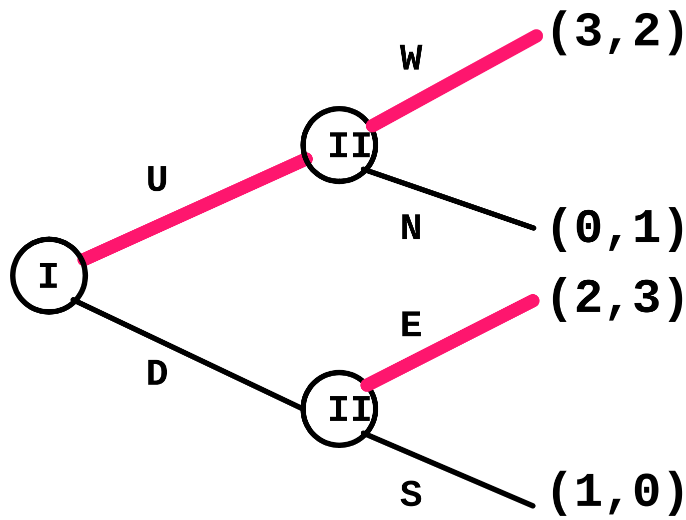

Figure 4.1: Comparing domination and backward induction.

Nevertheless we are able to apply the method of iterated elimination of weakly dominated strategies and identify the resulting irreducible games and the corresponding Nash equilibria. There are essentially two different orders in which we can eliminate weakly dominated strategies.[9] In all of these orders we always eliminate the strongly dominated strategy NS first:

Order A We eliminate all weakly dominated strategies immediately. This leaves player II with strategy WE and we identify the strategy profile (U, WE) as the resulting equilibrium.

Order B We first eliminate strategies NE and NS for player II. In the reduced game, player I has a dominant strategy in the form of strategy U. We thus arrive at an irreducible game consisting of the two Nash equilibria (U, WE) and (U, WS).

From the above we observe that both orders identify (U, WE) as a weakly iterated dominant equilibrium. But (U, WE) is also the backward induction solution in the extensive form representation of the game.

This is not a coincidence. Namely, both the method of backward induction and the method of iterated elimination of weakly dominated strategies work according to the same principle: *Both methods eliminate ways to play the game that lead to inferior payoffs.*

[9] In principle we can identify three different orders. Namely, besides the two or4ders identified below we can also argue that first eliminating strtategies NE and NS identifies a third order. But in the reduced game player I has no domination relationship between strategies U and D, thus reducing this third order to the Order A discussed in the text.

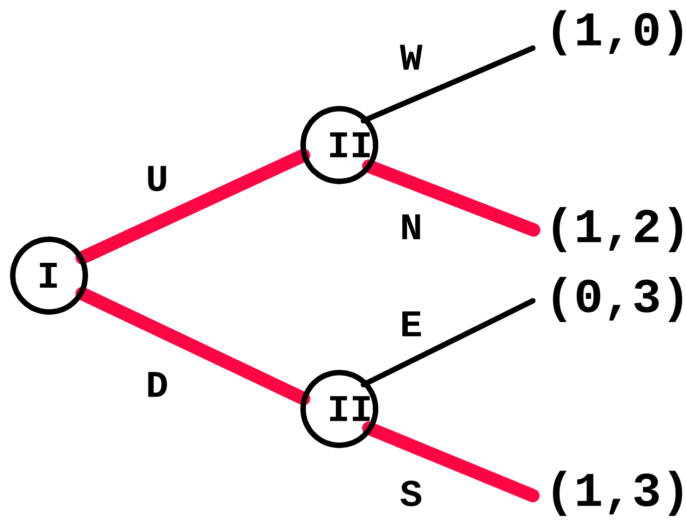

Figure 4.2: A game with multiple backward induction solutions.

In the method of backward induction this is accomplished by eliminating choices that lead to lower payoffs in every decision node in an iterated fashion, namely from the back of the game tree to the root node. In each step in the process of backward induction we eliminate "dominated" strategies in the decision nodes under consideration. This is very similar to eliminating dominated strategies in several steps in a matrix as is pursued in the method of iterated elimination of weakly dominated strategies.

To go back to the example given in Figure 4.1 and matrix (4.19), we notice that the application of the method of backward induction in the game tree identifies exactly the same equilibria as the method of iterated elimination of weakly dominated strategies executed according to Order A discussed above. Hence, backward induction corresponds to a *certain order* of iterated elimination of weakly dominated strategies, but not every arbitrary order. This might be explained by the fact that in a matrix we have lost the move order information that is captured by the tree representation of the game. Thus, if the order of elimination of weakly dominated strategies is exactly according to the move order identified in the game tree, then both methods lead to the same equilibria.

To conclude this section, I consider another example in which we apply both methods to make a comparison. In Figure 4.2 is depicted a game tree which has multiple solutions when we apply the method of backward induction. The identified paths from backward induction are shown in the game tree. Obviously we identify strategy profiles (U, NS) and (D, NS) as the solutions resulting from the application of the method of backward induction.

Next consider the corresponding ,strategic form representation of this game. In Figure 4.2 we identify $S_1 = \{U, D\}$, respectively $S_2 = \{WE, WS, NE, NS\}$, as the strategy sets for player I, respectively player II. We are now able to construct a 2×4-matrix representation of the strategic form of this game.

I \ **II**	**WE**	**WS**	**NE**	**NS**
U	0 1	0 1	2 1	2 1
D	2 0	3 1	2 0	3 1

(4.20)

In matrix (4.20) we identify four Nash equilibria: (U, NE), (U, NS), (D, WS), and (D, NS). We also identify strategy U as a weakly dominant strategy for player I and strategy NS as a weakly dominant strategy for player II. Hence, (U, NS) is an equilibrium in weakly dominant strategies in this game.

As before we identify three distinct orders of elimination of weakly dominated strategies:

Order A The most obvious way to proceed is to eliminate simultaneously all weakly dominated strategies for both players. This leads immediately to the equilibrium in weakly dominant strategies (U, NS).

Order B We might also first eliminate the weakly dominated strategy for player I. This leaves him with strategy U. In the reduced game we then eliminate strategies WE and WS for player II. This identifies two equilibria, (U, NE) and (U, NS).

Order C Conversely, we might first eliminate the weakly dominated strategies for player II. This leaves her with strategy NS. No further reduction of the reduced game tree is possible and we have identified equilibria (U, NS) and (D, NS).

From the above it becomes clear that backward induction in the game tree given in Figure 4.2 corresponds to the elimination of weakly dominated strategies according to *Order C* in the matrix (4.20).

I summarize the main conclusion of this section in the following property:

Proposition 4.13 *Consider a two-person game with complete information in its extensive form as well as in the corresponding strategic form. The application of the method of backward induction is equivalent to the execution of the method of iterated elimination of weakly dominated strategies in a certain order. This order of elimination is determined by the sequential structure represented in the game tree.*

4.4 Nash equilibrium and (common) knowledge

In the introduction to this chapter I state that Nash equilibrium is based on a simple, myopic type of rationality. In this section I, however, show that this simple rationality does not translate in a simple level of knowledge that is necessary to play the Nash equilibrium strategies. I will show with some examples that the knowledge requirements with regard to Nash equilibrium are indeed very demanding.

I consider three informational configurations with regard to the two-player game under consideration. These correspond to (1) those games in which there exists a unique equilibrium in dominant strategies, (2) those games in which one of the players has a dominant strategy, and (3) those games in which neither of the players has a dominant strategy. I will try to argue that the levels of "knowledge" used to play the equilibrium strategies in these three types of games is very different. Consequently the levels of reasoning and deduction used by the two players in the game differ very much for these three types of games.

First, let us consider two-person games in which *both players have a dominant strategy*. Thus, these games have a unique Nash equilibrium, which is equal to the unique equilibrium in dominant strategies. In Game 4.15 Player I has strategy D as his dominant strategy, while Player II has strategy R as her dominant strategy. The unique Nash equilibrium is the equilibrium in dominant strategies (D, R).

I \ **II**	**L**	**R**
U	1 0	2 0
D	2 2	3 1

Game 4.17

What are the informational requirements in Game 4.17 to establish this equilibrium? The answer is very simple: Both players only need to know their *own* payoffs. Namely, each player has a dominant strategy. Whatever the other player does, that strategy is always the best response. So, both players do not have to know anything about the other player's payoffs to play this game "correctly."

So, I conclude that in general there are actually no informational requirements whatsoever to play a game with dominant strategies correctly and to establish the unique equilibrium. I could say that both players require a "zeroth order knowledge" of the game in order to play this game "correctly."

Next consider a game in which *Player I has a dominant strategy, while Player II has no dominant strategy*. In Game 4.18 this is the case. Player I has strategy D as his dominant strategy, but Player II does not have a dominant strategy. However, Player II has certainly a unique best response to strategy D of Player I, namely strategy R. The unique Nash equilibrium in this game is therefore (D, R).

I \ **II**	**L**	**R**
U	3 0	2 0
D	2 2	3 1

Game 4.18

Again for Player I to play Game 4.18 "correctly" is very simple: Irrespective of the payoffs of his opponent, he always would choose his dominant strategy D. Thus, again Player I requires no knowledge to play this game.

Now consider Player II. She has to know the payoffs of Player I to determine that he has a dominant strategy D and that her best response to that dominant strategy is to play R. However, she has to know more than that. She also has to know that Player I knows that he has a dominant strategy. Thus, she has to know that her opponent is rational and has complete knowledge of his own payoffs. I call this a "first order knowledge."

I conclude that Player I needs a zeroth order knowledge, while Player II requires a first order knowledge to play Game 4.16 correctly and to establish the unique Nash equilibrium. It is clear that playing a game in which one of the players has a dominant strategy remains relatively simple.

Next let us consider a game in which neither of the two players has a dominant strategy, but with a unique (strict) Nash equilibrium. Thus, this unique Nash equilibrium is determined by mutual best responses. Consider Game 4.19 below. This game has a unique (strict) Nash equilibrium, namely (D, R). However, neither of the players has a dominant strategy. Moreover, note that there are no (weak) domination relationships between the different strategies for the two players at all.

I \ **II**	**L**	**R**
U	1 4	2 -1
M	3 3	2 0
D	2 2	3 1

Game 4.19

What knowledge on the part of the two players is necessary to establish the unique Nash equilibrium in Game 4.19? The answer is that both players require a much higher order of knowledge than considered thus far. Obviously, both players require first order knowledge: Both players need to know their opponent's payoffs and the knowledge that this opponent knows his own payoffs.

But that knowledge is not sufficient to determine the best response for both players. Indeed, Player I needs to know that Player II knows Player I's payoffs. Similarly for Player II. This could be called second order knowledge. Is this second order knowledge to determine and play the Nash equilibrium? Hardly, because both players need to determine the best response to the best response of their opponent. So, both players need to know more: Player I requires to know that Player II has second order knowledge of the game, i.e., Player I knows that Player II knows that Player I knows Player II's payoffs and is rational. Similarly for Player II. This could be called third order knowledge.

We have to continue reasoning in this fashion to determine that actually both players require all orders of knowledge to play the unique Nash equilibrium correctly. Or, in order to determine a best response to a best response Player I knows that Player II knows that Player I knows that Player II knows that.....and so on. Similarly for Player II: She knows that Player I knows that Player II knows that Player I knows that Player II knows that.....and so on. Having all orders of knowledge on part of both players is called *common knowledge* of the game. In conclusion, I can state that both players require common knowledge of Game 4.19 to play the game correctly and to establish the unique strict Nash equilibrium.

The requirement of common knowledge is tremendously demanding as one can imagine. It is infinitely more demanding than just knowing the strategies of all players as well as the payoffs in the game and that all players are rational. One has to know that all players know this; that all players know that all players know this; etcetera. That is an amazing quantity of information! If one reasons hard enough, it might well lead to the conclusion that one needs supernatural abilities to play a game "correctly."

4.5 Problems

Problem 4.1 *Let $x \geqq 0$ be some given nonnegative number. Consider the following game in strategic form:*

II ***I***	***N***	***E***	***S***	***W***
U	0 0	$1+x$ 0	2 0	2 1
M	x 2	7 $1+x$	4 3	1 2
D	8 1	$7-x$ 0	0 0	0 2

Solve for this game and the variable x the following questions:

1. *For which values of x does this game have a unique Nash equilibrium? Show your computations.*

2. *Are there any values of x for which this game has multiple Nash equilibria? Show your computations.*
   If your answer is "Yes" to this question, give those values of x for which the game has a maximum number of Nash equilibria. What is this maximum number of equilibria? Explain your answer.

3. *Are there any values of x for which this game does not have any Nash equilibria? Explain your answer and if necessary support with some computations.*

Problem 4.2 *Let $x > 0$ and $y > 0$ be two given positive numbers. Consider the following zero-sum two-person game:*

I \ ***II***	***L***	***M***	***R***
U	0 0	$-x$ x	y $-y$
S	y $-y$	$-y$ y	1 -1
D	1 -1	0 0	x $-x$

Solve the following problems for this game:

1. *What does the "Solution Theorem" state about this game? Explain exactly what is implied by this theorem.*

2. *Let $y > 0$ have a given value. For which values of $x > 0$ does this game have (i) a unique Nash equilibrium, (ii) multiple Nash equilibria, and (iii) no Nash equilibria? For each of these cases give explicit computations. (*Hint*: There are only three potential equilibria in this game, (U, R), (S, R), and (D, R). Why?)*

3. *From the answers to the previous question, you can in principle make a two-dimensional "map" of the space of all (x, y) pairs in which you can identify different areas in which the game has no Nash equilibria (Area N), in which the game has a unique Nash equilibrium (Area U), and in which the game has multiple Nash equilibria (Area M). Draw such a map.*

Problem 4.3 *Consider the Battle of the Sexes game as given in Game 4.4. Suppose our couple decides to break through their coordination dilemma — or "conflict" — by tossing a coin. Both promise to abide by the results from this coin toss: if "heads" comes up, they go*

to the romantic movie and if "tails" comes up they will go to the action movie. They use a single coin toss to resolve their decision problem.
Suppose that our couple's agreement to resolve their conflict with a coin toss is not enforceable. *But they go ahead and toss the coin, which comes up "tails." What do you predict will happen now? Explain why you reach a certain conclusion.*

Problem 4.4 *In 1943, Japanese Admiral Imamura was ordered to transport Japanese troops across the Bismarck Sea to New Guinea. US Admiral Kenney was ordered to intercept Imamura's fleet and bomb the troop transports with his carrier based bombers. Imamura has to choose among either taking a two-day route to New Guinea ("short"), taking a longer three day route ("long"), or sending half his ships on one route and half on the other ("50/50"). Kenney had to decide whether to send all of his planes to look for Imamura along the short route ("short"), send all his planes along the long route ("long"), or split them ("50/50") and search both routes simultaneously. Once Imamura set out, he could not recall his ships. On the other hand, Kenney's planes returned to their carriers at the end of the day and could be reassigned on the following day. The game payoff matrix is given in the table below. This matrix assumes Kenney's objective was to maximize the number of days Kenney's planes could bomb Imamura's ships, and Imamura objective was to minimize the number of day's of bombing.*

1 \ 2	***Short***	***Long***	***50/50***
Short	2, −2	2, −2	2, −2
Long	1, −1	3, −3	2, −2
50/50	1.5, −1.5	2.5, −2.5	2.5, −2.5

In this matrix Admiral Kenny is Player 1 and Admiral Imamura is Player 2.

1. *For each player, which strategies are (i) strongly dominant, (ii) strongly dominated, (iii) weakly dominant, and (iv) weakly dominated.*

2. *The payoff matrix defined is belongs to certain class of games. What are these games called? Find all the Nash equilibria of the above game.*

Problem 4.5 *Consider the following two-person game in strategic form:*

II ***I***	***L***	***M***	***R***
U	15 40	5 5	10 15
S	35 35	35 25	35 15
D	30 40	5 25	25 25

This game has certain specific features that we explore in the following questions:

1. *Identify the three Nash equilibria in this game.*
2. *Is this game a constant-sum game? Explain your answer in detail. With reference to the "Solution Theorem", what can you say about the "value" of this game?*
3. *Which of the Nash equilibria identified under (a) are equilibria in (weakly) dominant strategies? What is the meaning of the notion of "equilibrium in weakly dominant strategies"? Discuss the definition in your own words and apply your definition to the game under consideration.*
4. *Does this game have an equilibrium in weakly dominant strategies? If you think it has, give its strategy profile. If not, please explain why this game does not have such an equilibrium.*

Problem 4.6 *Consider the game given below:*

II ***I***	***L***	***R***
U	2 1	0 0
S	2 1	2 3
D	0 0	2 3

Solve the following questions:

1. *Are there any strictly dominated strategies?*
2. *Are there any weakly dominated strategies?*

3. *Find all the Nash equilibria.*

Problem 4.7 *The 31 members of the board of the Tusker Corporation are about to take a secret ballot on whether to accept the merger proposal of Mammoth Inc.. Each member can vote either to* ***accept*** *the proposal or* ***reject*** *the proposal or to* ***abstain****. In order for the proposal to be accepted, 16 board members must vote to accept. All 31 board members care about being on the winning side. That is, if the proposal is accepted, each member would prefer to have voted to accept it; and if the proposal is rejected, each member would prefer to have voted to reject it. Find the two Nash equilibria of this game.*

Problem 4.8 *Let $x \geqq 0$ be some nonnegative number. Given x consider the following two-person game:*

I \ ***II***	***X***	***Y***	***Z***	***T***	***U***
N	2, 4	x, 3	8, 2	1, 1	9, 3
W	3, 5	4, 3	5, 3	3, 3	9, 4
S	2, 0	3, 0	4, 4	6, 4	8, 5
E	1, 1	2, 1	3, 9	5, 5	7, 7

Solve the following problems for x:

1. *Find all Nash equilibria of this game for arbitrary values of x. For which values of x does the game have multiple Nash equilibria? For which values of x does it have no Nash equilibria?*

2. *Let $x < 4$. Apply the method of iterated elimination of (strongly) dominated strategies and derive the weakly irreducible game. Show that the weakly irreducible game identifies a unique iterated dominant equilibrium for these values of x.*

3. *Let $x > 4$. Again, apply the method of iterated elimination of (strongly) dominated strategies and derive the weakly irreducible game. Show that the weakly irreducible game identifies multiple Nash equilibria in the original game.*

4. *What can be said when $x = 4$? Explain.*

Problem 4.9 *Consider the following two-person game:*

A \ B	X	Y	Z	T	U
N	2, 3	3, 3	8, 2	1, 1	0, 2
W	1, 4	3, 3	5, 3	3, 3	8, 3
S	1, 8	3, 0	4, 4	5, 4	7, 3
E	1, 1	2, 1	3, 0	5, 5	7, 12

Solve the following problems for this game.

1. *Find all Nash equilibria in this game.*

2. *Which of the Nash equilibria identified is an equilibrium in weakly dominant strategies? Explain why there cannot be multiple equilibria in weakly dominant strategies.*

3. *Apply the method of iterated elimination of weakly dominated strategies. How many ways can you identify in which to execute this procedure? Identify which Nash equilibria are identified as iterated dominant equilibria through the different ways that you described.*

Problem 4.10 *Consider the following two-person game in strategic form:*

I \ II	L	M	R
U	0, 0	1, 3	1, 2
S	1, 2	1, 4	0, 1
D	1, 1	0, 0	1, 0

Show that there are three Nash equilibria in this game, but that there is only one of these three that is identified as the iterated weakly dominant equilibrium.

Problem 4.11 *Given is a two-person game. The payoffs are given in the matrix below:*

I \ ***II***	***X***	***Y***	***Z***	***T***	***U***
N	4 2	−1 5	0 1	3 5	2 0
W	1 1	5 1	4 −1	−1 4	0 5
S	0 2	0 3	1 1	−2 2	−1 2
E	0 1	1 2	0 −1	5 1	5 1

1. *Find all Nash equilibria of the game given above.*
2. *Show by iterated elimination of strongly dominated strategies that the given game does not have an equilibrium in dominant strategies. Derive the weakly irreducible game for this game.*
3. *Argue that given your answer in question 1 you should have expected the outcome in question 2. Explain in detail.*

Problem 4.12 *Given is a game in extensive form represented by the game tree given in Figure 4.3.*

1. *Explain why the game under consideration is a constant-sum game. What does the "Solution Theorem" predict about the Nash equilibria of this game?*
2. *Apply the method of backward induction to the game tree given in Figure 4.3. What is the unique backward induction solution that you find?*
3. *Give the normal form of the game tree given in Figure 4.3.*
4. *Find all Nash equilibria in the matrix derive under question 3. Show that the backward solution you identified in question 2 is among the Nash equilibria identified.*
5. *Show that thee is an equilibrium in weakly dominant strategies in the matrix that you constructed for question 3. Explain why there are similarities between the method of backward induction and the dominance structure in the strategic form representation of the game. In particular, explain why the equilibrium found in question 2 and the equilibrium in weakly dominant strategies are the same.*

Problem 4.13 *Consider the game tree given in Figure 4.4. Note that this is a game tree of a two-person game with incomplete information.*

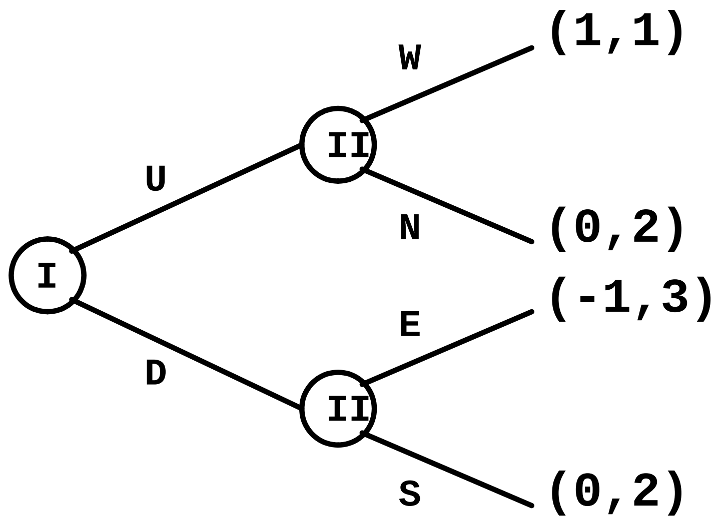

Figure 4.3: Game tree for Problem 4.12.

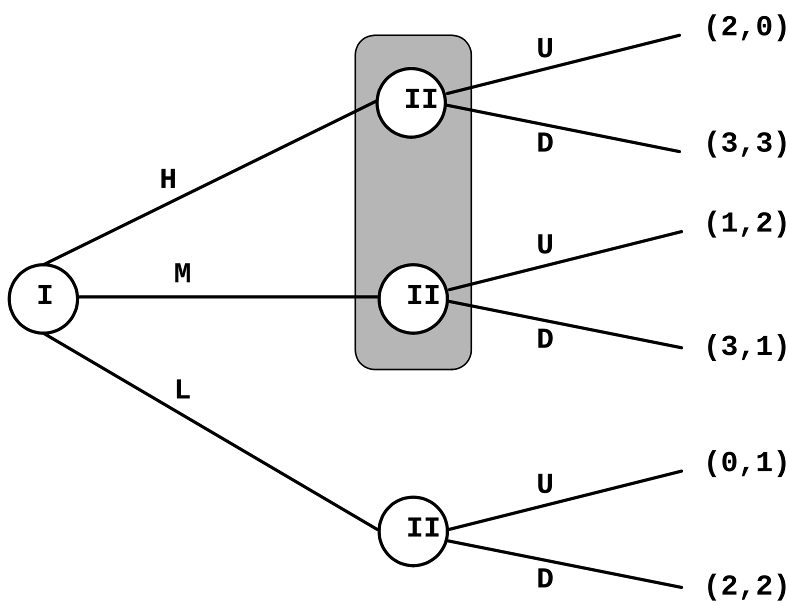

Figure 4.4: Game tree for Problem 4.13

1. *Give a short description of the method of "Backward Induction" and explain why this method cannot be applied straightforwardly to the game tree given.*

2. *Give a definition of the concept of a* strategy *in the context of a game tree. Explain your definition carefully and give your opinion of the usefulness of this concept in the analysis of a game.*

3. *Construct the strategic (or normal) form representation of the game tree given. Carefully construct the matrix representation of this strategic form game.*

4. *Find all Nash equilibria in the matrix corresponding to the normal form representation of the game. Is any of these Nash equilibria an equilibrium in weakly dominant strategies? Explain your answer and if necessary show your analysis.*

5. *Give a short description of the method of "iterated elimination of weakly dominated strategies" in a normal form game.*
   You can usually identify several different orders in which weakly dominated strategies can be eliminated. Identify two different orders in the game under consideration.
   Apply the method of iterated elimination of weakly dominated strategies in the two different orders that you identified to the matrix that you constructed in question 3. Identify precisely which equilibria you identify through application of this method and the different orders in which you apply this method.

Problem 4.14 *Consider the game tree given in Figure 4.5. Notice that in this game tree there are two nature nodes, $N1$ and $N2$, indicating risk elements in the game.*

1. *Explain in your own words the notion of the* Expected Utility Hypothesis *and how it applies to evaluate risky outcomes in games.*

2. *Apply the Expected Utility Hypothesis to construct a matrix representing the strategic form of the game tree given in the figure above.*

3. *Explain in your own words the difference between a regular Nash equilibrium and a* strict *Nash equilibrium. Using the matrix constructed in the previous question, show that the game has a unique Nash equilibrium and that this equilibrium is strict.*

Problem 4.15 *Consider a game in strategic form represented by the following matrix:*

II ***I***	***N***	***E***	***S***	***W***
U	6 1	2 5	5 1	3 0
M	2 3	1 1	3 2	2 2
D	4 0	5 0	1 1	0 5

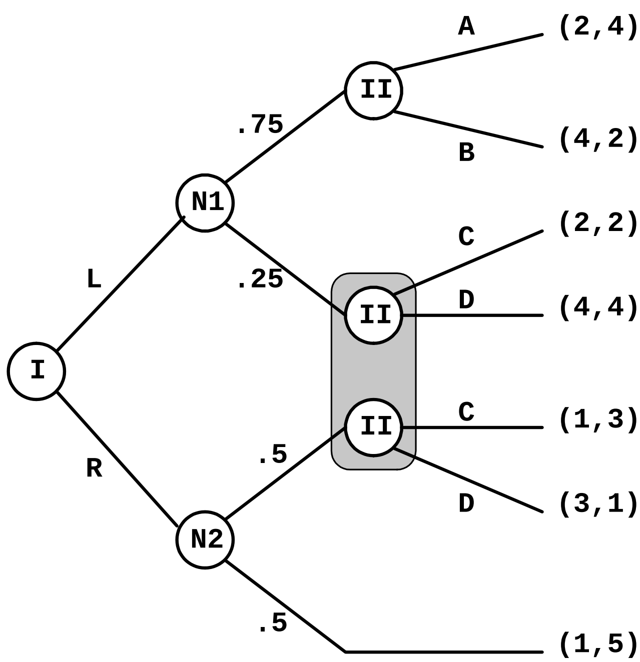

Figure 4.5: Game tree for Problem 4.14

1. *Give the definition of the property that one strategy* strongly dominates *another strategy for a certain player in a game. To illustrate your definition use an example from the strategic form game given above.*

2. *Apply the method of iterated elimination of strongly dominated strategies to reduce the game given above to an irreducible game. For each of the steps in this procedure you are asked to explicitly give the reduced game and to clearly label the strategies that you eliminate in each step of the procedure. Show that this irreducible game consists of a single field only, thus identifying an iterated dominant equilibrium.*

3. *Explain why any iterated dominant equilibrium has to be unique as well as a strict Nash equilibrium.*

Problem 4.16 *Consider the following game in normal form:*

B / ***A***	***N***	***L***	***M***	***K***
S	1 / 1	1 / 2	0 / 5	0 / 4
T	2 / 1	2 / 4	1 / 6	2 / 3
R	1 / 0	2 / 1	5 / 4	3 / 3
V	3 / 0	2 / 3	2 / 7	3 / 4

1. *Identify all Nash equilibria in this game.*

2. *Reduce this game by eliminating the (strongly) dominated strategies for both players in an iterated fashion. Show that we arrive in this fashion at a* 3×3 *matrix. Explain why it is the case that all Nash equilibria identified in question (a) have to be within this most reduced matrix.*

3. *Take the* 3×3 *matrix identified in the previous question and apply the method of iterated elimination of* weakly *dominated strategies. Show that there are three different orders to eliminate these strategies in an iterated fashion. Determine for each of these three orders the resulting equilibria.*

Problem 4.17 *Consider a game played by two players, called A and B and an impartial referee*[10]*. This game consists of two rounds.*
In the first round player A makes a choice between U *and* D*. If player A selects* D*, the referee communicates this selection to player B and the game proceeds to the next round. If player A*

[10] This referee is *not* a player in this game. He should considered to be "part of the rules."

selects U, however, the game enters a middle stage in which an impartial referee will flip a fair coin. If heads comes up, the referee announces to player B that player A has selected U. (This is indeed the truth.) If tails comes up, the referee announces to player B that player A has selected D. (This is, of course, not the truth.) Subsequently the game proceeds to the second round.
In the second round, player B selects between N and S. The payoffs of this game are given in the following table:[11]

B ***A***	***N***	***S***
U	*2* *4*	*0* *6*
D	*2* *0*	*6* *5*

Solve the following questions for this game.

1. *Give the game tree for this game. Recall that a game tree can contain information sets as well as nature nodes. Can this game tree be solved by the application of the backward induction method? Explain.*

2. *Convert the game tree that you developed under (a) into a normal form. Be careful to explain what the strategies for the two players are.*

3. *Show that this game has exactly two Nash equilibria. Identify these Nash equilibria in your normal form.*

4.6 Answer Keys for some problems

Answer to Problem 4.13.

1. Backward induction eliminates inferior choices in each decision node, starting from the back of the game tree, working your way up to the root node. This method can only be applied to game trees with complete information, in which each decision moment is described by a single decision node and not an information set consisting of multiple nodes.

2. For a particular player in a game, a strategy is defined as a listing of decisions at each information set (decision moment) in the game tree under control of that player. A strategy describes the strategic choices that a player makes in a game. The strategic form representation of a game thus summarizes only the strategic structure of the game.

[11]It should be emphasized that this table is *not* the normal form representation of this game!