Chapter 1 - Functions, Graphs, and Models

Objective a: Reviewing functions

A **relation** is a correspondence such that each member in the **domain** (x-value) corresponds to at least one member in the **range** (y-value).
A **function** is a correspondence such that each member in the **domain** (x-value) corresponds to exactly one member in the **range** (y-value).

All functions are relations, but not all relations are functions. In terms of an analogy of a machine, for every value we input to the function, we get exactly one output. If we get more than one output from a single input, then the relation is not a function. Think of a machine that produces chocolate chip cookies. If we input the ingredients x, we should get chocolate chip cookies as our output f(x), so our machine would be a function. We can even change x (the ingredients), and we would still get chocolate chip cookies (there is more than one recipe for chocolate chip cookies). Now suppose that there was a machine that when you input a particular set of ingredients x, sometimes you get chocolate chip cookies, but other times you get creamed spinach as your output y. That machine would not be a function since you are getting two different outputs from the same input. Also, keep in mind that $y = f(x)$.

Determine if the following are functions:

Ex. 1a $g(x) = x^2$

Solution:
For each value of x, we get exactly one value. Thus, g is a function. The domain of the function is $(-\infty, \infty)$ and the range is $[0, \infty)$.

Ex. 1b $f(x) = \pm x$

Solution:
For each value of x with the exception of 0, we get two values for f(x). If $x = 3$, then $f(3) = \pm 3$ which means $f(3) = 3$ or -3. Thus, f is not a function.

Ex. 2a $h(x) = 3x - 4$

Solution:
For each value of x, we get exactly one value. Thus, h is a function. The domain and range is $(-\infty, \infty)$.

Ex. 2b $x = 4$

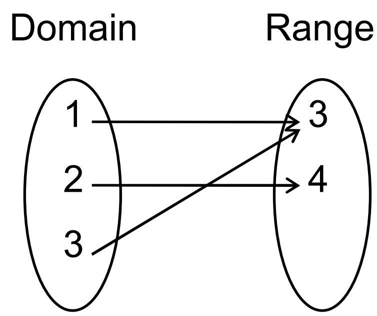
Solution:
For each value of x, we get an infinite number of values. Thus, $x = 4$ is not a function.

A function can also be given by a set of ordered pairs. In other words, we are given specifically what the function value (y) is for every input value (x).

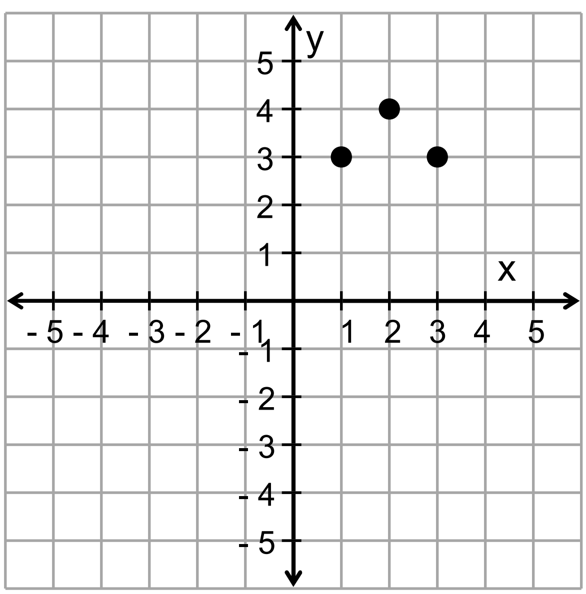
Ex. 3a $\{(1, 3), (2, 4), (3, 3)\}$

Ex. 3b $\{(3, 1), (4, 3), (3, 2)\}$

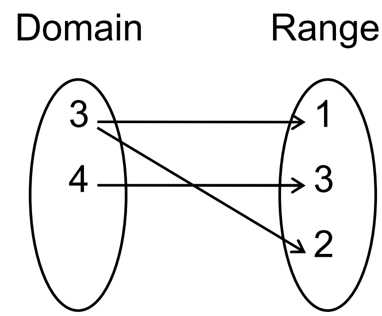
Solution:
{(1, 3), (2, 4), (3, 3)} tells us that f(1) = 3, f(2) = 4, and f(3) = 3. We can map this as follows:



Each member in the domain corresponds to exactly to one member in the range, so f is a function.
Note, the domain is {1, 2, 3} and the range is {3, 4}.
We can also plot these points on a graph:

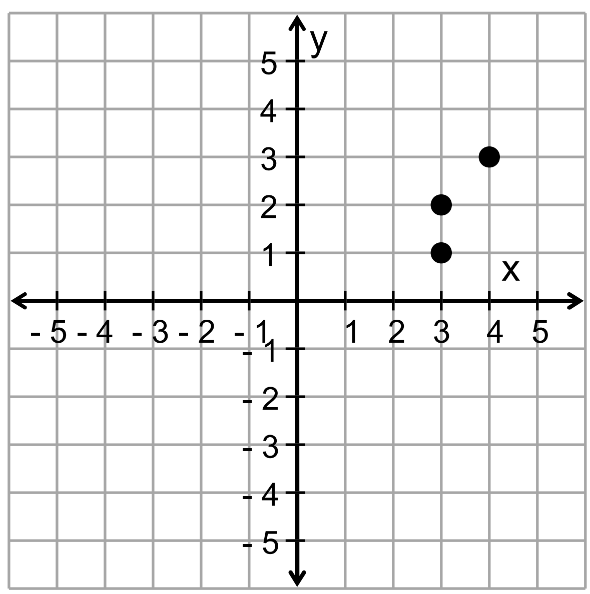


Solution:
{(3, 1), (4, 3), (3, 2)} tells us that f(3) = 1, f(4) = 3, and f(3) = 2. We can map this as follows:



The number 3 in the domain corresponds to both 1 and 2, so f is not a function.

Note, the domain is {3, 4} and the range is {1, 2, 3}.
We can also plot these points on a graph:



Notice that every vertical line intersects the graph of the function in 3a at most one time. Whereas there is a vertical line, namely x = 3, that intersects the graph of the relation in 3b twice. This means that a relation is not a function if there is a vertical line that intersects its graph more than once. This is known as the vertical line test.

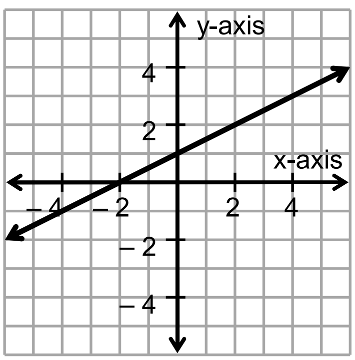
Objective b: Understanding and using the vertical line test.

Vertical line test for functions

If there exists a vertical line that passes through more than one point on the graph of f(x), then f(x) is not a function.

Given the graph below, a) Is it a function? b) If it is linear, write a down the linear equation using function notation if possible.
c) State the domain and range:

Ex. 4



Solution:

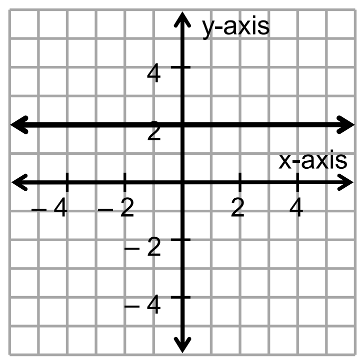
a) Yes, since every vertical line passes through at most one point.

b) From (– 2, 0) to (0, 1), rise 1 unit and run 2 units so, $m = \frac{\text{"rise"}}{\text{"run"}} = \frac{1}{2}$. The y-intercept is (0, 1), b = 1. Thus, $y = \frac{1}{2}x + 1$ or $f(x) = \frac{1}{2}x + 1$.

c) Domain: $\mathbb{R}$ or $(-\infty, \infty)$

Range: $\mathbb{R}$ or $(-\infty, \infty)$

Ex. 5



Solution:

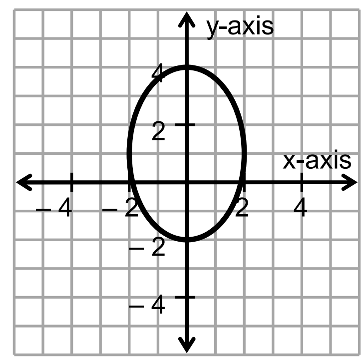
a) Yes, since every vertical line passes through at most one point.

b) The equation of a horizontal line is in the form f(x) = y = #, so the equation is so f(x) = 2.

c) Domain: $\mathbb{R}$ or $(-\infty, \infty)$

Range: {2}

Ex. 6



Solution:

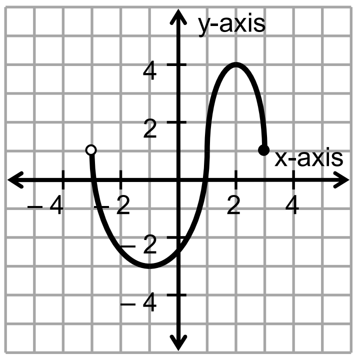
a) No, there is a vertical line (x = 1) that passes through two points.

b) This is not a linear equation. We will write N/A.

c) Domain: [– 2, 2] or $\{x \mid -2 \le x \le 2\}$

Range: [– 2, 4] or $\{y \mid -2 \le y \le 4\}$

Ex. 7



Solution:

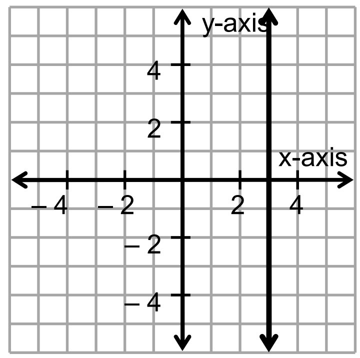
a) Yes, since every vertical line passes through at most one point.

b) This is not a linear equation. We will write N/A.

c) Domain: $(-3, 3]$ or $\{x \mid -3 < x \leq 3\}$

Range: $[-3, 4]$ or $\{y \mid -3 \leq y \leq 4\}$

Ex. 8



Solution:

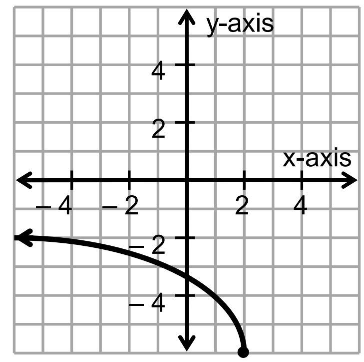
a) No, there is a vertical line ($x = 3$) that passes through an infinite number of points.

b) The equation of a vertical line is in the form x = #, so the equation is $x = 3$.

c) Domain $\{3\}$

Range: $(-\infty, \infty)$ or $\mathbb{R}$

Ex. 9



Solution:

a) Yes, since every vertical line passes through at most one point.

b) This is not a linear equation. We will write N/A.

c) Domain: $(-\infty, 2]$ or $\{x \mid x \leq 2\}$

Range: $[-6, -2]$ or $\{y \mid -6 \leq y \leq -2\}$

<u>Find the domain and range of the following:</u>

Ex. 10 $g(x) = 7x + 5$

Solution:

This is a linear equation that will extend on forever in both directions so, both the domain and range will be all real numbers:

Domain: $(-\infty, \infty)$ or $\mathbb{R}$ Range: $(-\infty, \infty)$ or $\mathbb{R}$

Ex. 11 $h(x) = \dfrac{7}{2x^2 + x - 15}$

Solution:

The only place we could get in trouble is if the denominator is 0.

Thus, $2x^2 + x - 15 \neq 0$

$(2x - 5)(x + 3) \neq 0$

$x \neq 2.5$ and $x \neq -3$

For the range, we can positive and negative values, but not zero since if $\dfrac{7}{2x^2 + x - 15} = 0$, then $7 = 0$ which is impossible.

Domain: $\{x \mid x \neq 2.5 \;\&\; -3\}$ or $(-\infty, -3) \cup (-3, 2.5) \cup (2.5, \infty)$

Range: $\{y \mid y \neq 0\}$ or $(-\infty, 0) \cup (0, \infty)$

Ex. 12 $f(x) = \sqrt[4]{3-2x}$

Solution:
For an even root function to be defined, the radicand has be greater than or equal to 0.
Thus, $3 - 2x \geq 0$
$- 2x \geq - 3$
$x \leq 1.5$
The range is all positive real numbers.
Domain: $(-\infty, 1.5]$
Range: $[0, \infty)$

Ex. 13 $g(x) = \ln(5x - 3)$

Solution:
For a logarithmic function to be defined, $5x - 3$ has to be greater than 0. Thus,
$5x - 3 > 0$
$5x > 3$
$x > 0.6$
The range is all real numbers.
Domain: $(0.6, \infty)$
Range: $(-\infty, \infty)$

Objective c: Evaluating functions and working with functions.

<u>Given $f(x) = -2x^2 + 3x - 1$, find and simplify:</u>

Ex. 14a $f(-3)$ Ex. 14b $f(0.2)$
Ex. 14c $f(3a)$ Ex. 14d $f(-7b)$
Ex. 14e f(pizza) Ex. 14f $f(x + h)$

Solution:
a) $f(-3) = -2(-3)^2 + 3(-3) - 1 = -18 - 9 - 1 = -28$
b) $f(0.2) = -2(0.2)^2 + 3(0.2) - 1 = -0.08 + 0.6 - 1 = -0.48$
c) $f(3a) = -2(3a)^2 + 3(3a) - 1 = -18a^2 + 9a - 1$
d) $f(-7b) = -2(-7b)^2 + 3(-7b) - 1 = -98b^2 - 21b - 1$
e) $f(\text{pizza}) = -2(\text{pizza})^2 + 3(\text{pizza}) - 1$
f) $f(x + h) = -2(x + h)^2 + 3(x + h) - 1$
$= -2(x^2 + 2xh + h^2) + 3(x + h) - 1$
$= -2x^2 - 4xh - 2h^2 + 3x + 3h - 1$

<u>Write a linear function:</u>

Ex. 15 Find the linear function passing through the point $(-2, 6)$
a) parallel to $g(x) = 3x - 2$.
b) perpendicular to $g(x) = 3x - 2$.

Solution:
a) Parallel lines have the same slope, so $m_{||} = 3$.
$y - y_1 = m(x - x_1)$ (point-slope form)
$y - 6 = 3(x - (-2)) = 3(x + 2) = 3x + 6$
$$\begin{array}{r} y - 6 = 3x + 6 \\ \underline{+ 6 = \quad + 6} \\ y = 3x + 12 \end{array} \Rightarrow \quad f(x) = 3x + 12$$

b) The slope of the line perpendicular is the negative reciprocal of 3. So, $m_{\perp} = -\frac{1}{3}$.

$y - y_1 = m(x - x_1)$ (point-slope form)

$y - 6 = -\frac{1}{3}(x - (-2)) = -\frac{1}{3}(x + 2) = -\frac{1}{3}x - \frac{2}{3}$

$$
\begin{array}{rcl}
y - 6 &=& -\frac{1}{3}x - \frac{2}{3} \\
+6 &=& \qquad + \frac{18}{3} \\
\hline
y &=& -\frac{1}{3}x + \frac{16}{3}
\end{array}
\quad \Rightarrow \quad f(x) = -\frac{1}{3}x + \frac{16}{3}
$$

Ex. 16 Find the linear function passing through the points (– 3, 9) and (2, – 1).

Solution:

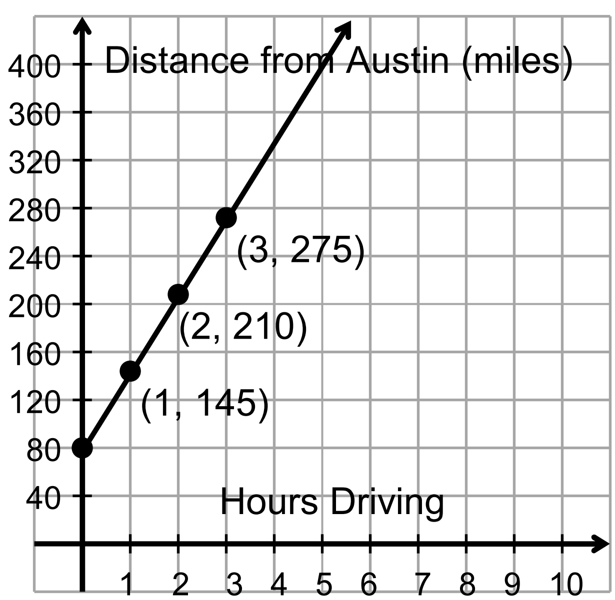
First, calculate the slope:

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-1 - 9}{2 - (-3)} = \frac{-10}{5} = -2, \text{ so } m = -2.$$

$y - 9 = -2(x - (-3)) = -2(x + 3) = -2x - 6$

$$
\begin{array}{rcl}
y - 9 &=& -2x - 6 \\
+9 &=& \qquad +9 \\
\hline
y &=& -2x + 3
\end{array}
\quad \Rightarrow \quad f(x) = -2x + 3
$$

Ex. 17 Given the graph below, a) find the average rate of change and b) find a linear function.



Solution:

a) The average rate of change is the slope of the line which corresponds to the speed of the car. We first pick two distinct points on the graph: (2, 210) & (3, 275).

Calculating the slope, we get: $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{275 - 210}{3 - 2} = \frac{65}{1} = 65$.

Notice that this is the same as his speed. Thus, the slope in this context describes how his distance changes with respect to time or more simply his speed. Thus, the average rate of change is 65 miles per hour.

b) Since m = 65 and the y-intercept is (0, 80), then we can write the equation in slope intercept form:

$y = mx + b$

$y = 65x + 80$

Using t in place of x and d(t) in place of y, our function becomes:

$d(t) = 65t + 80$.

Objective d: Applications of functions

Solve the following:

Ex. 18 Leroy invested \$3000 at 7.5% interest compounded quarterly. How much is in the account after 5 years?

Solution:

Recall the formula for compound interest: $A = P\left(1 + \frac{i}{n}\right)^{nt}$ where P is the principal, i is the interest converted to a decimal, n is the number of compoundings per year, and t is the number of years.

Since P = \$3000, i = 7.5% = 0.075, n = 4, and t = 5 years, then

$$A = P\left(1 + \frac{i}{n}\right)^{nt} = 3000\left(1 + \frac{0.075}{4}\right)^{4(5)} = 3000(1 + 0.01875)^{20}$$

$$= 3000(1.01875)^{20} = 3000(1.44994802571...) \approx \$4349.84$$

Ex. 19 The demand function for selling a certain type of can opener is given by $x = D(p) = (p - 4)^2$ where p is the price per unit and D(p) is the number of can openers (in thousands of can openers) demand by the consumers. The supply function for the same can opener is given by $x = S(p) = p^2 + 2p - 34$ where p is the price per unit and S(p) is the number of can openers (in thousands can openers) the sellers are willing to sell. Find the equilibrium point (p_E, x_E).

Solution:

The equilibrium point is where supply meets demand. To find it, we set D(p) = S(p):

$$
\begin{aligned}
(p - 4)^2 &= p^2 + 2p - 34 \\
p^2 - 8p + 16 &= p^2 + 2p - 34 \\
-p^2 - 2p \qquad &= -p^2 - 2p \\
\hline
-10p + 16 &= -34 \\
-16 &= -16 \\
\hline
-10p &= -50 \\
p &= \$5
\end{aligned}
$$

$D(5) = (5 - 4)^2 = 1$ thousand

The equilibrium point is (\$5, 1 thousand) or in words, at the equilibrium point, one thousand can openers will be sold at \$5 per can opener.

Objective e: Converting between rational exponents & radical expressions.

Recall that:

1) $a^{1/n} = \sqrt[n]{a}$ if $\sqrt[n]{a}$ is defined.

2) $a^{m/n} = \sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m$ if $\sqrt[n]{a}$ is defined.

3) $a^{-m/n} = \dfrac{1}{\sqrt[n]{a^m}} = \dfrac{1}{(\sqrt[n]{a})^m}$ if $\sqrt[n]{a}$ is defined and $a \neq 0$.

Write using rational exponents:

20a) $\sqrt[7]{d}$ 20b) $\sqrt[5]{x^4}$ 20c) $\dfrac{1}{\sqrt{c}}$ 20d) $\dfrac{1}{\sqrt[8]{a^5}}$

Solution:

a) $\sqrt[7]{d} = d^{1/7}$

b) $\sqrt[5]{x^4} = x^{4/5}$

c) $\dfrac{1}{\sqrt{c}} = \dfrac{1}{c^{1/2}} = c^{-1/2}$

d) $\dfrac{1}{\sqrt[8]{a^5}} = \dfrac{1}{a^{5/8}} = a^{-5/8}$

Write as a radical:

21a) $x^{1/9}$ 21b) $t^{5/3}$ 21c) $a^{-1/3}$

21d) $r^{-8/11}$ 21e) $64^{-4/3}$

Solution:

a) $x^{1/9} = \sqrt[9]{x}$

b) $t^{5/3} = \sqrt[3]{t^5}$

c) $a^{-1/3} = \dfrac{1}{a^{1/3}} = \dfrac{1}{\sqrt[3]{a}}$

d) $r^{-8/11} = \dfrac{1}{r^{8/11}} = \dfrac{1}{\sqrt[11]{r^8}}$

e) $64^{-4/3}\ \dfrac{1}{64^{4/3}} = \dfrac{1}{(\sqrt[3]{64})^4} = \dfrac{1}{(4)^4} = \dfrac{1}{256}$

Objective g: Writing functions as a composition of two simpler functions.

Find f(x) and g(x) such that the given function h(x) = (f o g)(x):

22a) $h(x) = \sqrt{x-7}$

22b) $h(x) = 5(2x-7)^2$

22c) $h(x) = \dfrac{3}{x+6}$

22d) $h(x) = \ln(x^3 - 5x + e^x)$

Solution:

a) The outside function f is the square root function and the inside function g is $x - 7$. Thus, $f(x) = \sqrt{x}$ and $g(x) = x - 7$.

b) The outside function f is the quadratic function and the inside function g is $2x - 7$. Thus, $f(x) = 5x^2$ and $g(x) = 2x - 7$.

c) The outside function f is the inverse function and the inside function g is $x + 6$. Thus, $f(x) = \dfrac{3}{x}$ and $g(x) = x + 6$.

d) The outside function f is the natural logarithmic function and the inside function g is $x^3 - 5x + e^x$. Thus, $f(x) = \ln(x)$ and $g(x) = x^3 - 5x + e^x$.

Objective h: Review of graphing functions.

Graphing the following:

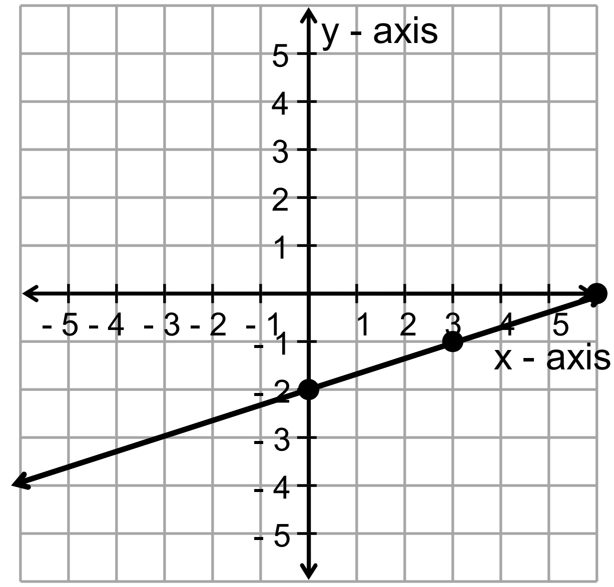
Ex. 23 $2x - 6y = 12$

Solution:

First solve the equation for y:

$$2x - 6y = 12$$
$$\underline{-2x \qquad = -2x}$$
$$\frac{-6y}{-6} = \frac{-2x + 12}{-6}$$
$$y = \frac{1}{3}x - 2$$

The slope is $\frac{1}{3} = \frac{\text{"rise"}}{\text{"run"}}$ and the y-intercept is (0, – 2). Plot the point (0, – 2). Then from that point rise 1 unit and run 3 units to get another point. From that new point, rise another 1 unit and run 3 more units to get the third point. Now, draw the graph.



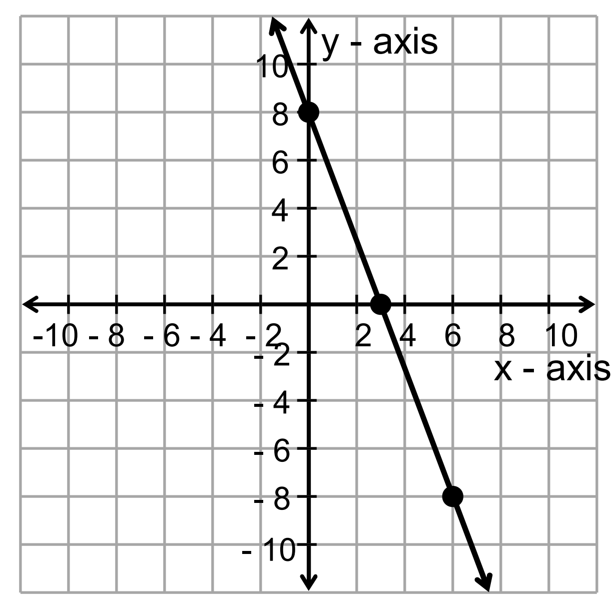
Ex. 24 $8x + 3y = 24$

Solution:

First solve the equation for y:

$$8x + 3y = 24$$
$$\underline{-8x \qquad = -8x}$$
$$\frac{3y}{3} = \frac{-8x + 24}{3}$$
$$y = -\frac{8}{3}x + 8$$

The slope is $-\frac{8}{3} = \frac{-8}{3} = \frac{\text{"rise"}}{\text{"run"}}$ and the y-intercept is (0, 8). Plot the point (0, 8). Then from that point fall 8 units and run 3 units to get another point. From that new point fall another 8 units and run 3 more units to get the third point. Now, draw the graph.



Ex. 25 $f(x) = \frac{x^2 - 4x + 3}{x - 3}$

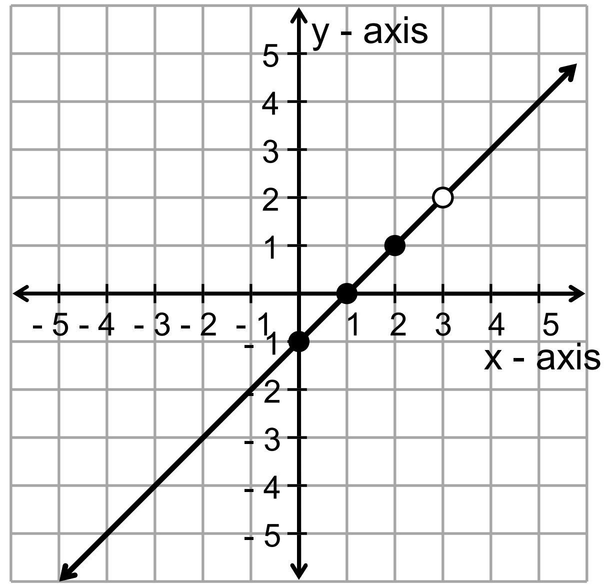
<u>Solution:</u>

The domain of f is all real numbers except $x \neq 3$. First, simplify the rational expression :

$$f(x) = \frac{x^2 - 4x + 3}{x - 3} = \frac{(x-3)(x-1)}{x-3}$$

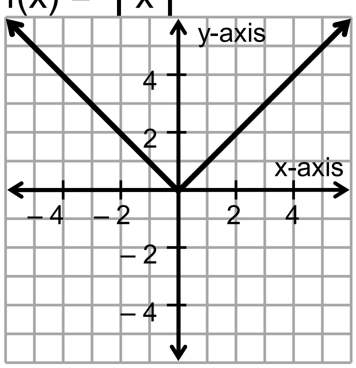
$$= x - 1 \text{ for } x \neq 3$$

The graph of f will be the line $y = x - 1$ except there will be a "hole" at $x = 3$ since $x = 3$ is not in the domain of f. The y-intercept is $(0, -1)$ and the slope is 1.

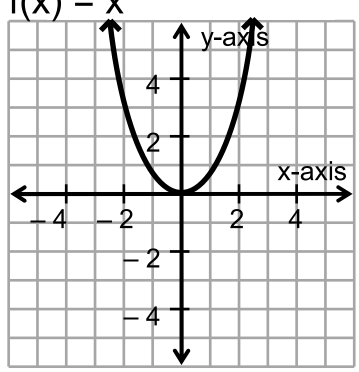


Recall the graphs of basic functions:

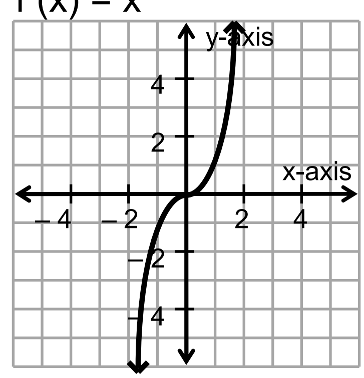
$f(x) = |x|$



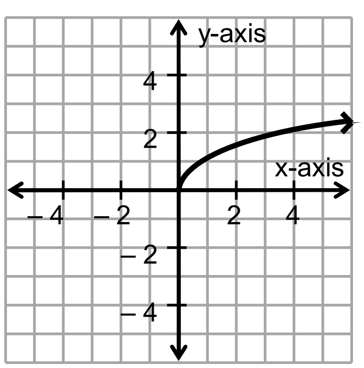
$f(x) = x^2$



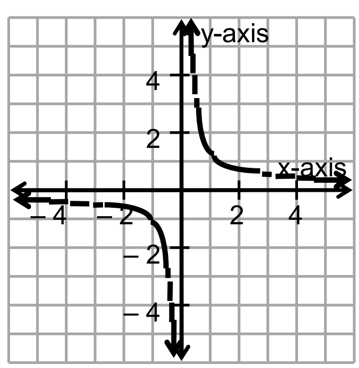
$f(x) = x^3$



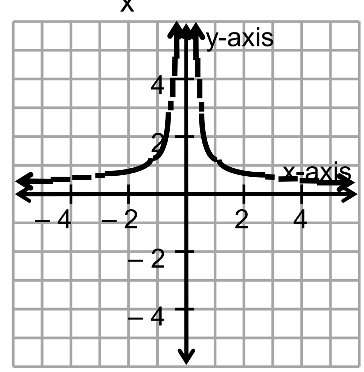
$f(x) = \sqrt{x}$



$f(x) = \frac{1}{x}$



$f(x) = \frac{1}{x^2}$



General Strategy for transformations:

In graphing $f(x) = a \bullet f(x - h) + k$, we will start with the graph of f(x) &

i) Stretch it by a factor of a if $|a| > 1$ or shrink it by a factor of a if $|a| < 1$.

ii) Reflect it across the x-axis if a is negative.

iii) Shift it horizontally by h units and vertically by k units.

Ex. 26 Graph $h(x) = \frac{2}{3}|x + 3| - 2$

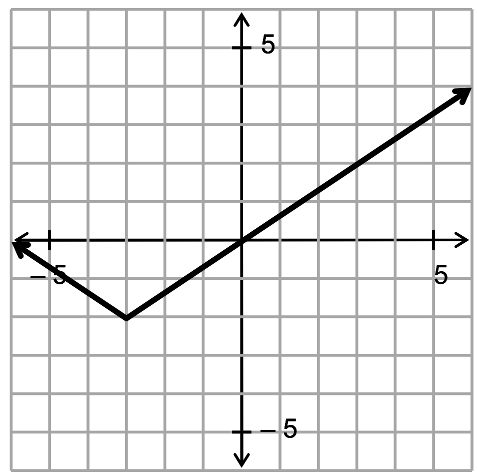
Solution:

Let's go through the steps of our general strategy:

i) Since $|a| = \frac{2}{3}$, the graph is shrunk to $\frac{2}{3}$ of its size.

ii) Since a is positive, the graph is not reflected across the x-axis.

iii) Since k is – 2 and h is – 3, the graph is shifted down by 2 units and to the left by 3 units.



Ex. 27 Graph $f(x) = -2|x + 1/2| + 4$

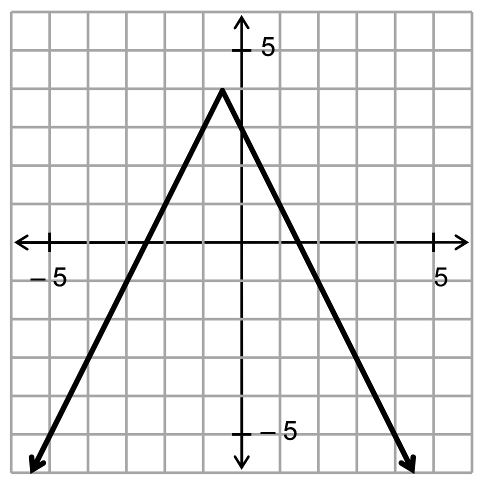
Solution:

Let's go through the steps of our general strategy:

i) Since $|a| = 2$, the graph is stretched by a factor of 2.

ii) Since a is negative, the graph is reflected across the x-axis.

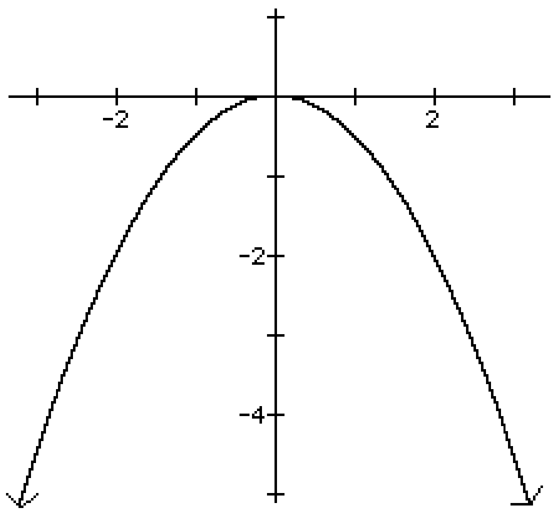
iii) Since k is 4 and h is – 1/2, the graph is shifted up by 4 units and to the left by 1/2 unit.



Ex. 28 Graph $h(x) = -0.5x^2$

Solution:

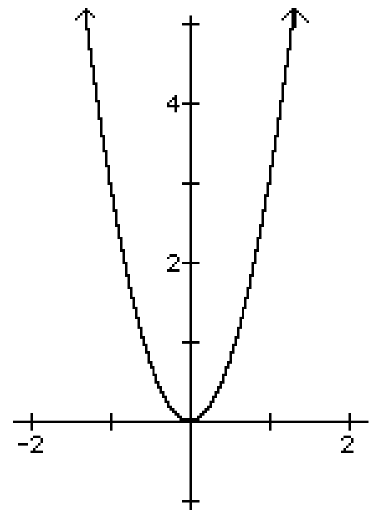
Since a is – 0.5, the graph of x^2 is shrunk to half its size and reflected across the x-axis.



Ex. 29 Graph $g(x) = 3x^2$

Solution:

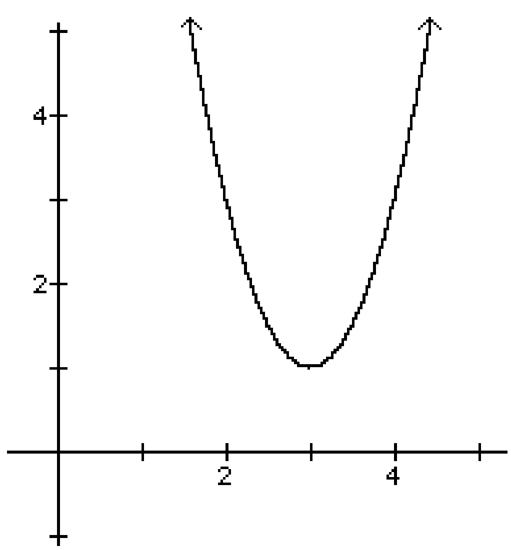
Since a is 3, the graph of x^2 is stretched by a factor of 3.



Ex. 30 Graph $h(x) = 2(x - 3)^2 + 1$

Solution:

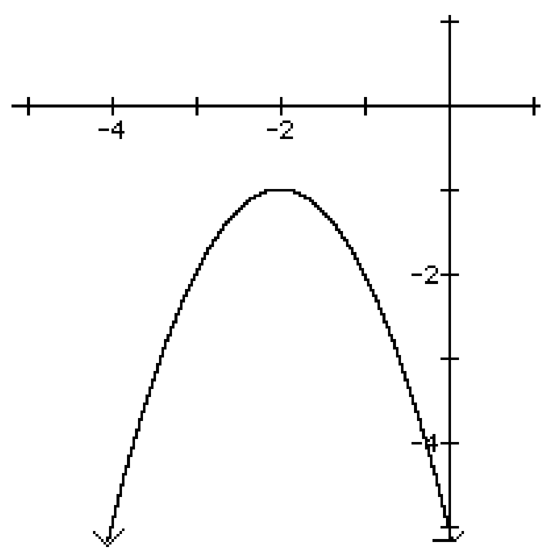
Since a = 2, the graph is stretched by a factor of 2. Since k is 1 and h is 3, the graph is shifted up 1 unit and to the right by 3 units.



Ex. 31 Graph $f(x) = -(x + 2)^2 - 1$

Solution:

Since a = – 1, the graph is reflected across the x - axis. Since k = – 1 and h = – 2, the graph is shifted down 1 unit & to the left by 2 units.



Note, if the quadratic function f(x) is not already in graph form, find the vertex by using $h = \frac{-b}{2a}$ and k = f(h).

Find the vertex, write the equation in graphing form and find the intercepts:

Ex. 32 $f(x) = 2x^2 - 12x + 19$

Solution:

a) Since a = 2, b = – 12, and c = 19, then

$h = \frac{-b}{2a} = \frac{-(-12)}{2(2)} = \frac{12}{4} = 3$

and k = f(h) = f(3)

$= 2(3)^2 - 12(3) + 19$

= 18 – 36 + 19 = 1.

The vertex is (3, 1).

b) Since a = 2, h = 3 & k = 1, then the graphing form is

$f(x) = a(x - h)^2 + k = 2(x - 3)^2 + 1$

or $f(x) = 2(x - 3)^2 + 1$

c) To find the y-intercept, let x = 0:

$f(0) = 2(0)^2 - 12(0) + 19$

= 19

To find the x-intercept, set f(x) = 0 and solve:

$2x^2 - 12x + 19 = 0$

Use the Quadratic Formula:

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$= \frac{-(-12) \pm \sqrt{(-12)^2 - 4(2)(19)}}{2(2)}$

$= \frac{12 \pm \sqrt{144 - 152}}{4}$ which is not real.

Hence:

y-int: (0, 19)

x-int: None

Ex. 33 $g(x) = -x^2 - 3x + 4$

Solution:

a) Since a = – 1, b = – 3 and c = 4, then

$h = \frac{-b}{2a} = \frac{-(-3)}{2(-1)} = \frac{3}{-2} = -1.5$

and k = g(h) = g(– 1.5)

$= -(-1.5)^2 - 3(-1.5) + 4$

= – 2.25 + 4.5 + 4 = 6.25.

The vertex is (– 1.5, 6.25)

b) Since a = – 1, h = – 1.5, & k = 6.25, then the graphing form is $g(x) = a(x - h)^2 + k$

or $g(x) = -(x + 1.5)^2 + 6.25$

c) To find the y-intercept, let x = 0:

$g(0) = -(0)^2 - 3(0) + 4$

= 4

To find the x-intercept, set g(x) = 0 and solve

$-x^2 - 3x + 4 = 0$

$-(x^2 + 3x - 4) = 0$

$-(x + 4)(x - 1) = 0$

x = – 4 or x = 1

Hence:

y-int: (0, 4)

x-int: (– 4, 0) & (1, 0)

Graph the following piece-wise defined function:

34)

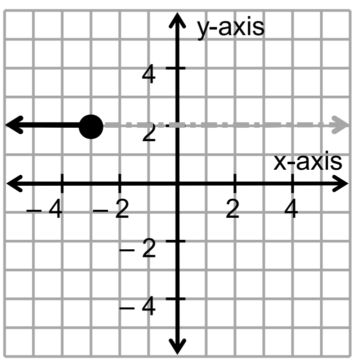
$$f(x) = \begin{cases} 2, & x \le -3 \\ x + 5, & -3 < x < 0 \\ x^2, & x \ge 0 \end{cases}$$

Solution:

First, graph each piece:

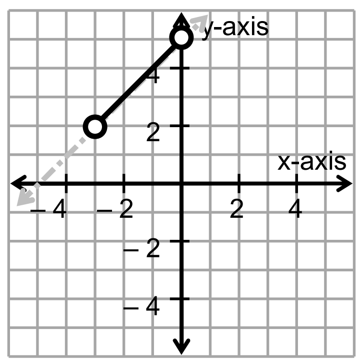
$x \le -3$

$f(x) = 2$



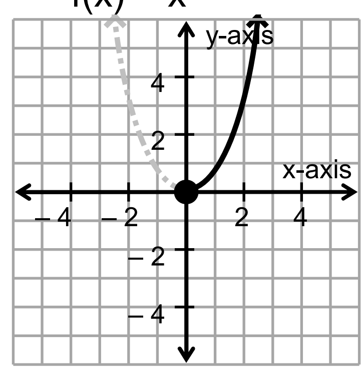
$-3 < x < 0$

$f(x) = x + 5$



$x \ge 0$

$f(x) = x^2$



Now, put the pieces together on one graph:

f(x)