Sect 2.1 & 2.2 – Limits and Continuity

Objective a: Understanding limits numerically and graphically.

In this section, we need to explore a powerful idea in Calculus called limits. The limit process involves examining the behavior of a function f(x) (think of the y - values) as x approaches a number c. The number c itself may or may not be in the domain of f. Let's consider an example:

Ex. 1 For the function $f(x) = \frac{x^2 - x - 2}{x - 2}$,

a) Determine its domain.
b) Evaluate f at x = 1.8, 1.9, 1.99, 1.999, 2.001, 2.01, 2.1, and 2.2.
c) Describe the behavior of f (i.e., the y - values) as x approaches 2.

Solution:

a) The function is defined so long as the denominator does not equal zero which will happen if x = 2. So, the domain of f is all real numbers except 2 or $(-\infty, 2) \cup (2, \infty)$.

b) Let's create a table for the given values:

x	1.8	1.9	1.99	1.999	2	2.001	2.01	2.1	2.2
f(x)	2.8	2.9	2.99	2.999	—	3.001	3.01	3.1	3.2

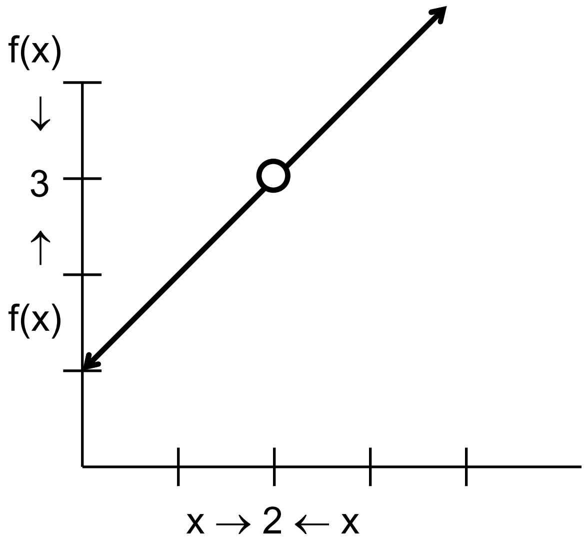
c) The table suggests that as x approaches the number 2, f(x) approaches the number 3. **Thus, the limit of f(x) as x approaches 2 is equal to 3.**

The way we state this mathematically is: $\lim_{x \to 2} f(x) = 3$.

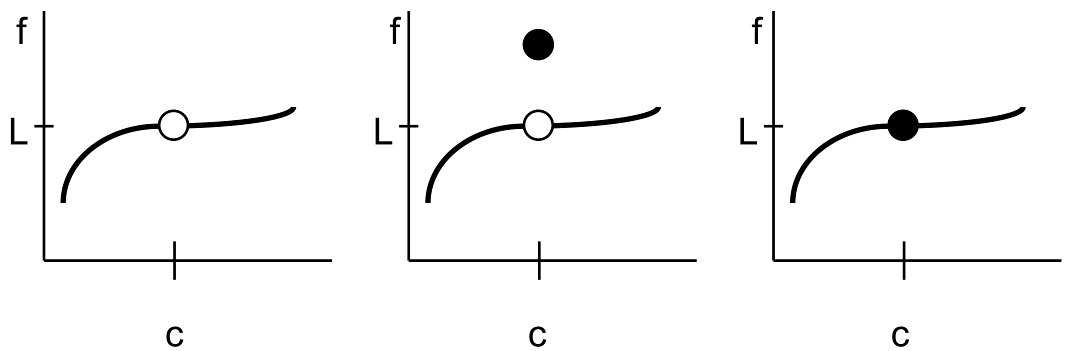
Keep in mind that this describes the behavior of f *near* x = 2 and *not at* x = 2. The function itself is undefined at x = 2. In general, if f(x) gets closer to L as x gets closer to c from either side of c, then

$$\lim_{x \to c} f(x) = L$$

In the first example, the graph of f(x) looks line a straight line with a hole at x = 2:



In general, there are three cases with the behavior of a function at x = c.



$\lim_{x \to c} f(x) = L$, but f(c) does not exist.

$\lim_{x \to c} f(x) = L$, but $f(c) \neq L$.

$\lim_{x \to c} f(x) = L$ and f(c) = L.

It is important to realize that in all of the examples above, the limit exists and is equal to L since as x approaches c from either direction, f(x) approaches L. We can talk about left-hand and right-hand limits. If x approaches c from the left side or the negative side (x < c), then we would denote that as:

$$\lim_{x \to c^-} f(x) = L$$

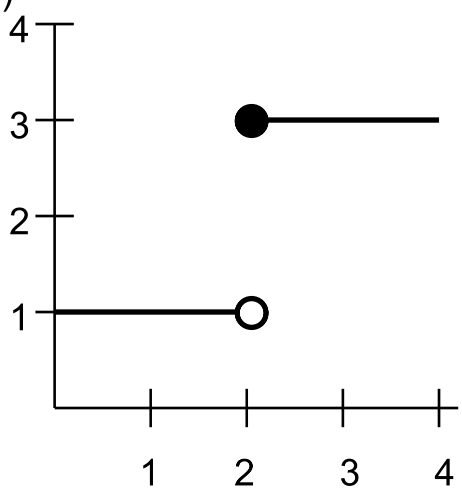
If x approaches c from the right side or the positive side (x > c), then we would denote that as:

$$\lim_{x \to c^+} f(x) = L$$

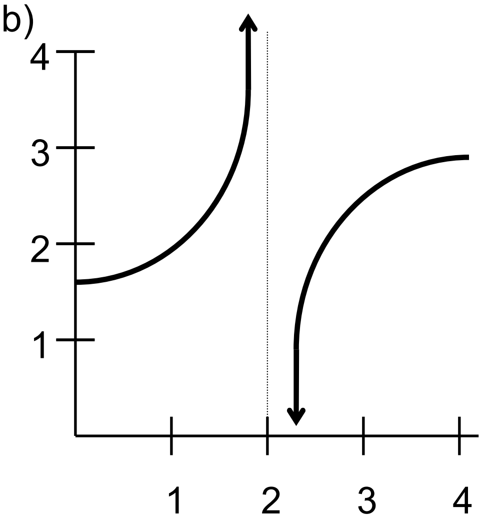
Let's look at a couple of examples where the limit does not exist.

Ex. 3

a)



b)



In this example, the $\lim\limits_{x \to 2^-} f(x) = 1$ and $\lim\limits_{x \to 2^+} f(x) = 3$. Since $\lim\limits_{x \to 2^-} f(x) \neq \lim\limits_{x \to 2^+} f(x)$, then $\lim\limits_{x \to 2} f(x)$ does not exist.

In this example, the $\lim\limits_{x \to 2^-} f(x) = \infty$ and $\lim\limits_{x \to 2^+} f(x) = -\infty$. Since $\lim\limits_{x \to 2^-} f(x) \neq \lim\limits_{x \to 2^+} f(x)$, then $\lim\limits_{x \to 2} f(x)$ does not exist.

The symbol ∞ (infinity) is not an actual number, but it represents a quantity that is larger than any finite bound. In other words, as x approaches 2 from the left side, f(x) increasing without bound. Likewise with $-\infty$, as x approaches 2 from the right side, f(x) decreasing without bound.
In both examples above, the left-hand and right-hand limits exist, but they are not the same value. For the limit to exist, the left-hand and right-hand limits have to be the same.

Objective b: Finding Limits Algebraically

Properties of limits:

Let $\lim\limits_{x\to c} f(x) = L$, $\lim\limits_{x\to c} g(x) = M$, k and p be constants, then

1) $\lim\limits_{x\to c} [f(x) \pm g(x)] = \lim\limits_{x\to c} f(x) \pm \lim\limits_{x\to c} g(x) = L \pm M$

2) $\lim\limits_{x\to c} k\bullet f(x) = k\bullet \lim\limits_{x\to c} f(x) = k\bullet L$

3) $\lim\limits_{x\to c} [f(x)\bullet g(x)] = [\lim\limits_{x\to c} f(x)]\bullet[\lim\limits_{x\to c} g(x)] = L\bullet M$

4) $\lim\limits_{x\to c} \frac{f(x)}{g(x)} = \frac{\lim\limits_{x\to c} f(x)}{\lim\limits_{x\to c} g(x)} = \frac{L}{M}$, provided $\lim\limits_{x\to c} g(x) = M \neq 0$.

5) $\lim\limits_{x\to c} [f(x)]^p = [\lim\limits_{x\to c} f(x)]^p = L^p$, provided $[\lim\limits_{x\to c} f(x)]^p$ exists.

6) $\lim\limits_{x\to c} k = k$

7) $\lim\limits_{x\to c} x = c$

Let p(x) and q(x) be polynomials, then

8) $\lim\limits_{x\to c} p(x) = p(c)$

9) $\lim\limits_{x\to c} \frac{p(x)}{q(x)} = \frac{p(c)}{q(c)}$, provided $q(c) \neq 0$.

Ex. 4 Evaluate the following limits:

a) $\lim\limits_{x\to -2} x^2 - 3x + 2$

b) $\lim\limits_{x\to 1} \frac{x^2-9}{x^2+16}$

c) $\lim\limits_{x\to 3} \frac{x^3 - 27}{x^2 - 9}$

Solution:

a) $\lim\limits_{x\to -2} x^2 - 3x + 2 = (-2)^2 - 3(-2) + 2 = 4 + 6 + 2 = 12.$

b) $\lim\limits_{x\to 1} \frac{x^2-9}{x^2+16} = \frac{(1)^2-9}{(1)^2+16} = \frac{1-9}{1+16} = \frac{-8}{17}.$

c) $\lim\limits_{x\to 3} \frac{x^3 - 27}{x^2 - 9} = \frac{3^3 - 27}{3^2 - 9} = \frac{27-27}{9-9} = \frac{0}{0}$, indeterminate. It appears that we cannot work part c, but remember the limit asks what is the behavior of f(x) when x is <u>near</u> 3, but not at 3.

Let's see if we can simplify the expression before taking the limit:

$$\lim_{x\to 3}\frac{x^3-27}{x^2-9} = \lim_{x\to 3}\frac{(x-3)(x^2+3x+9)}{(x-3)(x+3)} = \lim_{x\to 3}\frac{\cancel{(x-3)}(x^2+3x+9)}{\cancel{(x-3)}(x+3)}$$

$$= \lim_{x\to 3}\frac{x^2+3x+9}{x+3} = \frac{(3)^2+3(3)+9}{(3)+3} = \frac{9+9+9}{3+3} = \frac{27}{6} = \frac{9}{2}.$$

This suggests a strategy for finding limits:

1) Try replacing x by c.
2) If that does not work, try simplifying the expression and then replace x by c.
3) If that fails, look at the left-hand and right-hand limits and see if they are the same.

Ex. 5 Evaluate $\lim_{x\to 2}\frac{3x^2-4x-4}{5x^2-4x-12}$.

Solution:

We first try plugging in 2 for x.

$\lim_{x\to 2}\frac{3x^2-4x-4}{5x^2-4x-12} = \frac{3(2)^2-4(2)-4}{5(2)^2-4(2)-12} = \frac{12-8-4}{20-8-12} = \frac{0}{0}$ which does not yield an answer. So, let's try simplifying:

$$\lim_{x\to 2}\frac{3x^2-4x-4}{5x^2-4x-12} = \lim_{x\to 2}\frac{(x-2)(3x+2)}{(x-2)(5x+6)} = \lim_{x\to 2}\frac{\cancel{(x-2)}(3x+2)}{\cancel{(x-2)}(5x+6)}$$

$$= \lim_{x\to 2}\frac{3x+2}{5x+6} = \frac{3(2)+2}{5(2)+6} = \frac{8}{16} = \frac{1}{2}.$$

Ex. 6 Evaluate $\lim_{x\to -5}\frac{x^2+10x+25}{x^2+25}$.

Solution:

We first try replacing x by – 5.

$$\lim_{x\to -5}\frac{x^2+10x+25}{x^2+25} = \frac{(-5)^2+10(-5)+25}{(-5)^2+25} = \frac{25-50+25}{25+25} = \frac{0}{50} = 0.$$

Since we got an answer, then the limit is 0.

Ex. 7 Evaluate $\lim\limits_{x \to 9} \frac{\sqrt{x}-3}{x-9}$.

Solution:
We first try plugging in 9 for x.

$\lim\limits_{x \to 9} \frac{\sqrt{x}-3}{x-9} = \frac{\sqrt{9}-3}{9-9} = \frac{0}{0}$ which does not yield an answer.

Next we will try to simplify the expression by rationalizing the numerator. To do that, we need to multiply top and bottom of the rational expression by the conjugate of $\sqrt{x} - 3$:

$$\lim_{x \to 9} \frac{\sqrt{x}-3}{x-9} = \lim_{x \to 9} \frac{\sqrt{x}-3}{x-9} \bullet \frac{\sqrt{x}+3}{\sqrt{x}+3} = \lim_{x \to 9} \frac{x-9}{(x-9)(\sqrt{x}+3)}$$

$$= \lim_{x \to 9} \frac{\cancel{x-9}}{\cancel{(x-9)}(\sqrt{x}+3)} = \lim_{x \to 9} \frac{1}{\sqrt{x}+3} = \frac{1}{\sqrt{9}+3} = \frac{1}{6}.$$

Ex. 8 Evaluate $\lim\limits_{x \to 2} f(x)$ where $f(x) = \begin{cases} 3x + 2, & x < 2 \\ x^2, & x \geq 2 \end{cases}$

Solution:
Simply plugging in does not work since the function "changes" at x = 2. We cannot simplify it algebraically, so we need to look at the left-hand and right-hand limits.

$\lim\limits_{x \to 2^-} f(x) = \lim\limits_{x \to 2^-} 3x + 2 = 3(2) + 2 = 8.$

$\lim\limits_{x \to 2^+} f(x) = \lim\limits_{x \to 2^+} x^2 = (2)^2 = 4.$ Since the left-hand and right-hand limits are not the same value, then $\lim\limits_{x \to 2} f(x)$ does not exist.

Ex. 9 Evaluate $\lim\limits_{x \to 4} \frac{x^2-16}{x^2-8x+16}$.

Solution:
We first try plugging in 4 for x:

$\lim\limits_{x \to 4} \frac{x^2-16}{x^2-8x+16} = \frac{(4)^2-16}{(4)^2-8(4)+16} = \frac{16-16}{16-32+16} = \frac{0}{0}$ which does not yield an answer. Next, we will try to simplify the expression:

$$\lim_{x\to 4}\frac{x^2-16}{x^2-8x+16} = \lim_{x\to 4}\frac{(x-4)(x+4)}{(x-4)(x-4)} = \lim_{x\to 4}\frac{\cancel{(x-4)}(x+4)}{\cancel{(x-4)}(x-4)} = \lim_{x\to 4}\frac{x+4}{x-4}$$

$= \frac{(4)+4}{(4)-4} = \frac{8}{0}$ which is undefined. This means that either the limit does not exist, or it is ∞ or $-\infty$. To find out, we need to examine the left-hand and right-hand limits:

$$\lim_{x\to 4^-}\frac{x+4}{x-4} = \frac{8}{\text{A small } - \# \to 0} = -\infty.$$

$\lim\limits_{x\to 4^+}\frac{x+4}{x-4} = \frac{8}{\text{A small } + \# \to 0} = \infty$. Since the left-hand and right-hand limits are not the same value, then

$\lim\limits_{x\to 4}\frac{x^2-16}{x^2-8x+16}$ does not exist.

Ex. 10 Evaluate $\lim\limits_{x\to -3}\frac{-1}{(x+3)^2}$.

Solution:

We first try plugging in – 3 for x:

$\lim\limits_{x\to -3}\frac{-1}{(x+3)^2} = \frac{-1}{((-3)+3)^2} = \frac{-1}{(0)^2} = \frac{-1}{0}$ which is undefined.

This means that either the limit does not exist, or it is ∞ or $-\infty$. To find out, we need to examine the left-hand and right-hand limits:

$$\lim_{x\to -3^-}\frac{-1}{(x+3)^2} = \frac{-1}{\text{A small} + \# \to 0} = -\infty.$$

$\lim\limits_{x\to -3^+}\frac{-1}{(x+3)^2} = \frac{-1}{\text{A small} + \# \to 0} = -\infty$. Since the left-hand and right-hand limits are equal to $-\infty$, then

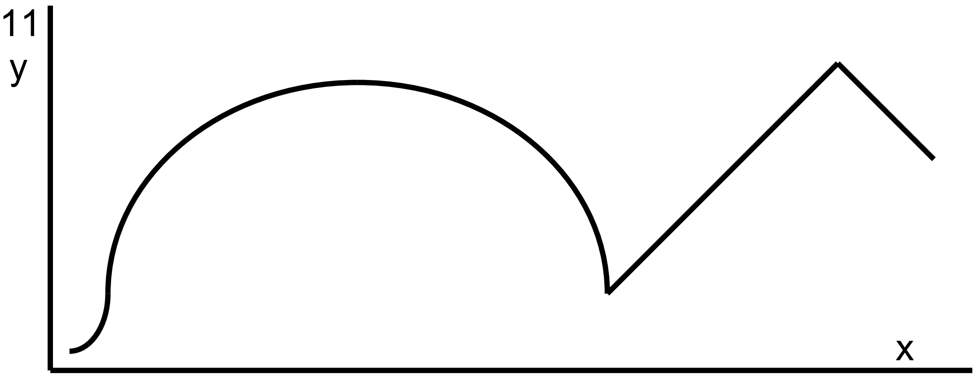
$$\lim_{x\to -3}\frac{-1}{(x+3)^2} = -\infty.$$

We will discuss limits at ∞ and $-\infty$ in more depth in chapter 3.

Objective c: Understanding Continuity

Continuity is an extremely important idea in mathematics. When we say that a function is continuous, it means that its graph has no holes or gaps. In other words, a function f is continuous at a point c if you can draw the graph through the point (c, f(c)) without lifting your pencil.

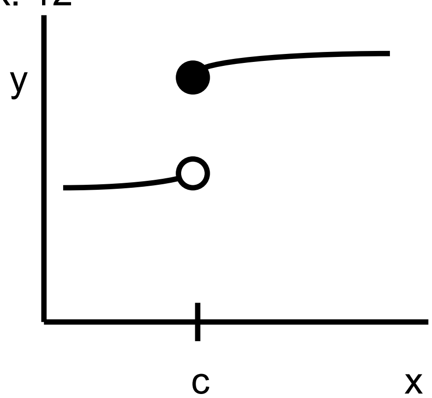
Ex. 11



Here, the function is continuous at every point since there are no holes or gaps in the graph.

Ex. 12

a)



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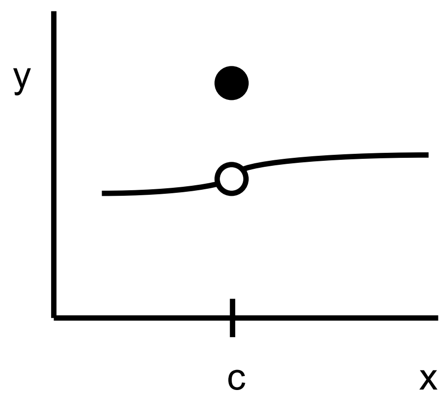
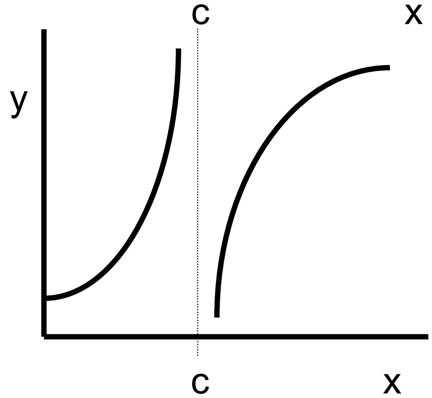


c)



In each of these graphs, the function is not continuous or discontinuous at x = c. In part c, f(c) is not defined. In part a, $\lim_{x \to c} f(x)$ does not exist. In part b, $\lim_{x \to c} f(x) \neq f(c)$. This suggests a more formal definition of continuity:

Definition: A function f is continuous at x = c if **all** the following are true:

1) f(c) is defined.
2) $\lim_{x \to c} f(x)$ exists.
3) $\lim_{x \to c} f(x) = f(c)$.

Ex. 13 Is $f(x) = 7x^2 - 6x + 3$ continuous at $x = -2$?

Solution:

1) $f(-2) = 7(-2)^2 - 6(-2) + 3 = 28 + 12 + 3 = 43$.
So $f(-2)$ is defined.
2) $\lim_{x \to -2} f(x) = \lim_{x \to -2} 7x^2 - 6x + 3 = 7(-2)^2 - 6(-2) + 3$
$= 28 + 12 + 3 = 43$. So, $\lim_{x \to -2} f(x)$ exists.
3) $\lim_{x \to -2} f(x) = 43 = f(-2)$.

Thus, the function is continuous at $x = -2$.

In general, if p and q are polynomials, then $\lim_{x \to c} p(x) = p(c)$ and $\lim_{x \to c} \frac{p(x)}{q(x)} = \frac{p(c)}{q(c)}$ provided that $q(c) \neq 0$. Also, $\lim_{x \to c} \sqrt[n]{f(x)} = \sqrt[n]{f(c)}$ provided that $\sqrt[n]{f(c)}$ is defined and c is not an endpoint of its domain. This means that polynomial and rational functions are continuous at every point in their domain and radical functions are continuous at every point in their domain except at the endpoints of the domain.

Ex. 14 State all the values where the following functions are continuous.

a) $f(x) = x^3 - 2x^2 + x - 5$

Solution:

Since f is a polynomial, it is defined for all real numbers and hence, it is continuous on $(-\infty, \infty)$.

b) $g(x) = \frac{x-5}{x^2+16}$

Solution:
Since g is a rational function, it is continuous on its domain. Since $x^2 + 16 \neq 0$, it is defined for all real numbers and hence, it is continuous on $(-\infty, \infty)$.

c) $h(x) = \frac{x+4}{x^2-16}$

Solution:
Since h is a rational function, it is continuous on its domain. Setting $x^2 - 16 = 0$ and solving yields:

$x^2 - 16 = 0$
$(x - 4)(x + 4) = 0$
$x = 4$ or $x = -4$.

Thus, the domain of h is $(-\infty, -4) \cup (-4, 4) \cup (4, \infty)$.
So, h is continuous everywhere except at $x = 4$ and -4.

d)
$$f(x) = \begin{cases} x^2 - 5, & x < 3 \\ 2x - 2, & 3 \leq x < 5 \\ x, & x \geq 5 \end{cases}$$

Solution:
Since each piece is a polynomial, the function is continuous for all values of x except possibly the values where the function "changes" ($x = 3$ and $x = 5$). Let's look at $x = 3$ first:

1) $f(3) = 2(3) - 2 = 6 - 2 = 4$.
So f(3) is defined.

2) $\lim\limits_{x\to 3^-} f(x) = \lim\limits_{x\to 3^-} x^2 - 5 = (3)^2 - 5 = 9 - 5 = 4$.
$\lim\limits_{x\to 3^+} f(x) = \lim\limits_{x\to 3^+} 2x - 2 = 2(3) - 2 = 6 - 2 = 4$. So,
$\lim\limits_{x\to 3} f(x)$ exists.

3) $\lim\limits_{x\to 3} f(x) = 4 = f(3)$.

So, f is continuous at $x = 3$.
Now, we will look at $x = 5$:

1) $f(5) = (5) = 5$.
So f(5) is defined.

2) $\lim_{x \to 5^-} f(x) = \lim_{x \to 5^-} 2x - 2 = 2(5) - 2 = 10 - 2 = 8.$

$\lim_{x \to 5^+} f(x) = \lim_{x \to 5^+} x = (5) = 5.$

Since $\lim_{x \to 5^+} f(x) \neq \lim_{x \to 5^-} f(x)$, $\lim_{x \to 5} f(x)$ does not exist.

So, f is not continuous at x = 5.
Therefore, f is continuous on $(-\infty, 5) \cup (5, \infty)$.

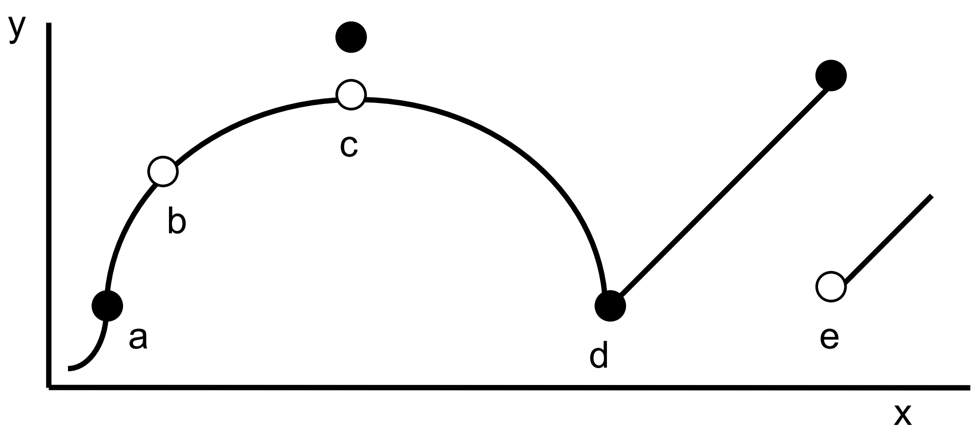
e) $g(x) = \sqrt[4]{x}$

Solution:
The domain of g is $[0, \infty)$. Zero is the endpoint of the domain of g. Since radical expressions are continuous at every point in their domain except at the endpoints, then g is continuous on $(0, \infty)$.

f) $f(x) = \sqrt[3]{x}$

Solution:
The domain of f is $(-\infty, \infty)$. Since there are no endpoints, then f is continuous on $(-\infty, \infty)$.

Ex. 15 Given the graph of f, at which of the labeled points is f continuous?



Solution:
The function f is only continuous at points a and d. It is discontinuous at b since f(b) is not defined. It is discontinuous at c since $\lim_{x \to c} f(x) \neq f(c)$. It is discontinuous at x = e since $\lim_{x \to e} f(x)$ does not exist.

Ex. 16 Use the graph of f to the right to answer the following:

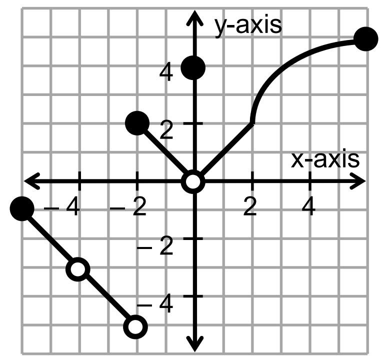
a) Find the domain.

b) Find the range.

c) Find $\lim_{x \to -4} f(x)$.

d) Find f(– 4).

e) Find $\lim_{x \to -2} f(x)$.



f) Find f(– 2).

g) Find $\lim_{x \to 0} f(x)$.

h) Find f(0)

i) For what values of x is f continuous?

<u>Solution:</u>

a) f is defined for all real numbers from – 6 to 6 inclusively except at x = – 4. Thus, the domain is [– 6, – 4) U (– 4, 6].

b) There are several gaps in the outputs from – 5 to 5 inclusively: – 5 and – 3 are missing and the interval (– 1, 0] is missing. Thus, the range is (– 5, – 3) U (– 3, – 1] U (0, 5].

c) Since $\lim_{x \to -4^-} f(x) = -3$ and $\lim_{x \to -4^+} f(x) = -3$, then $\lim_{x \to -4} f(x) = -3$.

d) f(– 4) is undefined.

e) Since $\lim_{x \to -2^-} f(x) = -5$ and $\lim_{x \to -2^+} f(x) = 2$, then $\lim_{x \to -2} f(x)$ = does not exist.

f) f(– 2) = 2

g) Since $\lim_{x \to 0^-} f(x) = 0$ and $\lim_{x \to 0^+} f(x) = 0$, then $\lim_{x \to 0} f(x) = 0$.

h) f(0) = 4.

i) f is continuous on [– 6, – 4) U (– 4, – 2) U (– 2, 0) U (0, 6].