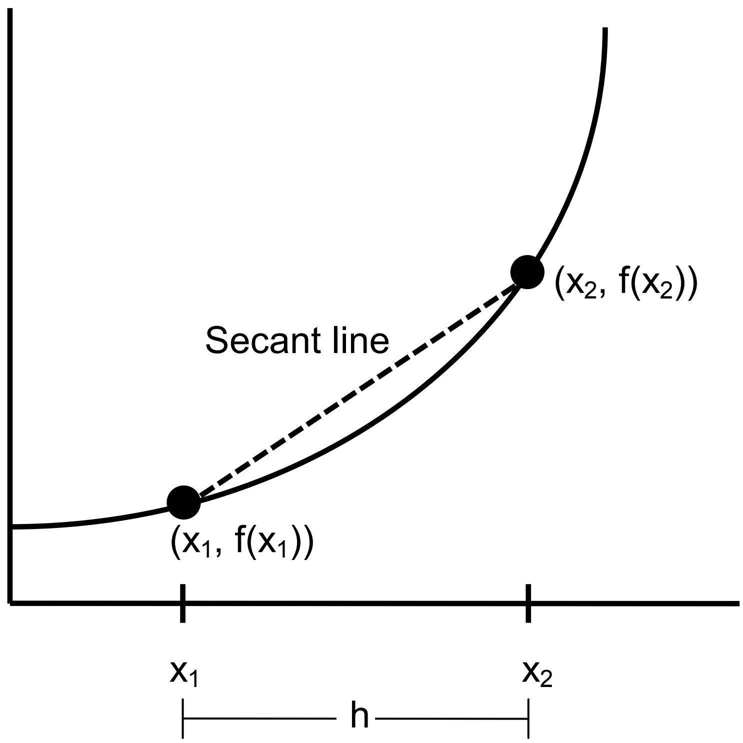
Section 2.3 – Average Rate of Change

For any two distinct points on a curve, we can approximate the rate of change from the first point to the second point by drawing a straight line (secant line) through those points and calculating the slope of that line. The name of the game in this chapter is to then push these points together and calculate the instantaneous rate of change at a point on the curve. Consider the following:



The slope of the secant line is $m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$

(The symbol delta, Δ, means "change in"). Let $h = x_2 - x_1$ and let $x = x_1$. Then $x_2 = x_1 + h = x + h$. Plugging into m, we get:

$$m = \frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{f(x+h) - f(x)}{h}.$$

The quotient, $\frac{f(x+h) - f(x)}{h}$, is called the **difference quotient**. Keep in mind that this is another formula for the slope of the line between two points on the curve.

Ex. 1 Given that $f(x) = x^2$, find the difference quotient when

a) $x = 2$ and $h = 0.1$.

b) $x = 2$ and $h = 0.01$.

c) $x = 2$ and $h = 0.001$.

d) $x = 2$ and $h \to 0$.

Solution:

a) $\frac{f(x+h)-f(x)}{h} = \frac{f(2+0.1)-f(2)}{0.1} = \frac{f(2.1)-f(2)}{0.1} = \frac{(2.1)^2-(2)^2}{0.1}$
$= \frac{4.41-4}{0.1} = \frac{0.41}{0.1} = 4.1$. So, the slope of the line or the average rate of change is 4.1.

b) $\frac{f(x+h)-f(x)}{h} = \frac{f(2+0.01)-f(2)}{0.01} = \frac{f(2.01)-f(2)}{0.01} = \frac{(2.01)^2-(2)^2}{0.01}$
$= \frac{4.0401-4}{0.01} = \frac{0.0401}{0.01} = 4.01$. So, the slope of the line or the average rate of change is 4.01.

c) $\frac{f(x+h)-f(x)}{h} = \frac{f(2+0.001)-f(2)}{0.001} = \frac{f(2.001)-f(2)}{0.001}$
$= \frac{(2.001)^2-(2)^2}{0.001} = \frac{4.004001-4}{0.001} = \frac{0.004001}{0.001} = 4.001$. So, the slope of the line or the average rate of change is 4.001.

d) As $h \to 0$, the value of the difference quotient appears to approach 4.

Ex. 2 Given that $f(x) = 3x^3$, find the difference quotient when

a) $x = 2$ and $h = 0.1$.

b) $x = 2$ and $h = 0.01$.

c) $x = 2$ and $h = 0.001$.

d) $x = 2$ and $h \to 0$

Solution:

a) $\frac{f(x+h)-f(x)}{h} = \frac{f(2+0.1)-f(2)}{0.1} = \frac{f(2.1)-f(2)}{0.1} = \frac{3(2.1)^3-3(2)^3}{0.1}$
$= \frac{27.783-24}{0.1} = \frac{3.783}{0.1} = 37.83$. So, the slope of the line or the average rate of change is 37.83.

b) $\frac{f(x+h)-f(x)}{h} = \frac{f(2+0.01)-f(2)}{0.01} = \frac{f(2.01)-f(2)}{0.01} = \frac{3(2.01)^3-3(2)^3}{0.1}$

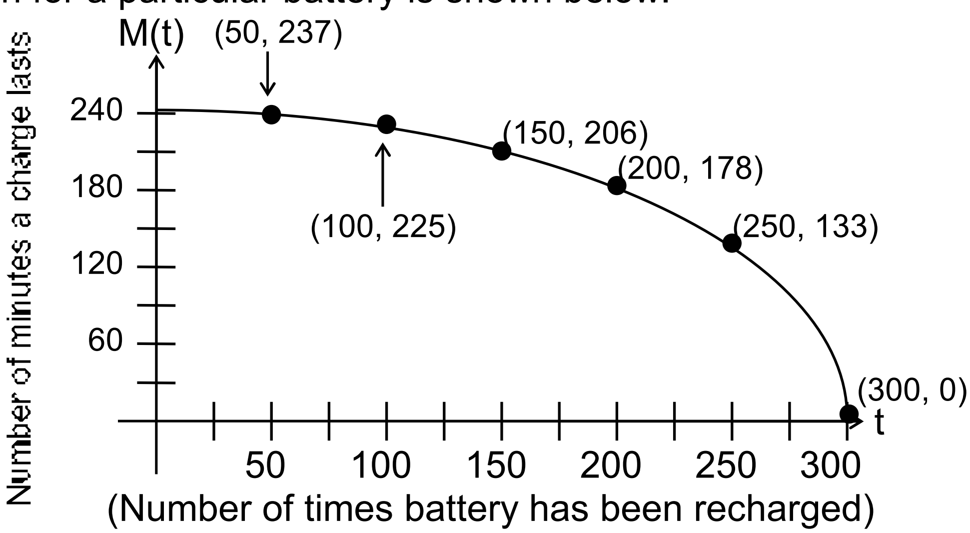
$= \frac{24.361803 - 24}{0.01} = \frac{0.361803}{0.01}$ = 36.1803. So, the slope of the line or the average rate of change is 36.1803.

c) $\frac{f(x+h) - f(x)}{h} = \frac{f(2+0.001) - f(2)}{0.001} = \frac{f(2.001) - f(2)}{0.001}$

$= \frac{3(2.001)^3 - 3(2)^3}{0.1} = \frac{24.036018003 - 24}{0.001} = \frac{0.0036018003}{0.001} =$ 36.018003. So, the slope of the line or the average rate of change is 36.018003.

d) As h $\rightarrow$ 0, the value of the difference quotient appears to approach 36.

Ex. 3 The number of minutes M(t) a laptop battery will last after being fully recharged is a function of the number of times (t) the battery has been recharged. The graph for a particular battery is shown below.



a) Find the average rate of change of M as t changes from 0 to 50, from 50 to 100, from 100 to 150, from 150 to 200, from 200 to 250, and from 250 to 300.

b) Find the average rate of change of M as t changes from 0 to 300. Does this accurately reflect how much charge the battery is losing over time?

Solution:

a) From 0 to 50: $\frac{237 - 240}{50 - 0} = \frac{-3}{50} = -0.06$.

So, the battery is losing an average of 0.06 minutes per charge.

From 50 to 100: $\frac{225-237}{100-50} = \frac{-12}{50} = -0.24$.

So, the battery is losing an average of 0.24 minutes per charge.

From 100 to 150: $\frac{206-225}{150-100} = \frac{-19}{50} = -0.38$.

So, the battery is losing an average of 0.38 minutes per charge.

From 150 to 200: $\frac{178-206}{200-150} = \frac{-28}{50} = -0.56$.

So, the battery is losing an average of 0.56 minutes per charge.

From 200 to 250: $\frac{133-178}{250-200} = \frac{-45}{50} = -0.9$.

So, the battery is losing an average of 0.9 minutes per charge.

From 250 to 300: $\frac{0-133}{300-250} = \frac{-133}{50} = -2.66$.

So, the battery is losing an average of 2.66 minutes per charge.

b) From 0 to 300: $\frac{0-240}{300-0} = \frac{-240}{300} = -0.8$.

So, the battery is losing an average of 0.8 minutes per charge. No, this does not accurately reflect how much charge the battery is losing over time. The loss of charge is much smaller early in the battery's life, but the loss of charge increases throughout the life of battery. The calculated average rate of change from 0 to 300 does not reflect this information.