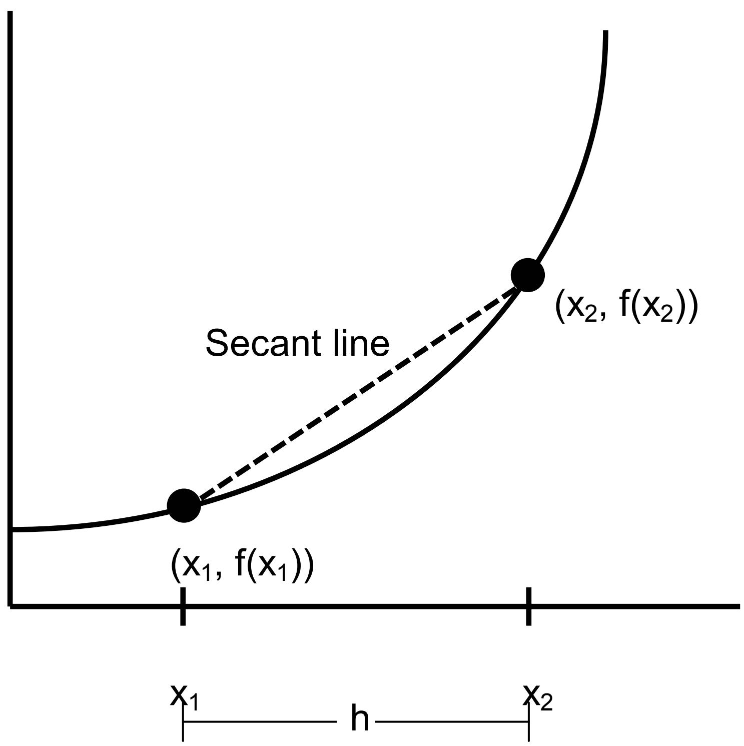
Section 2.4 – Differentiation Using Limits

Objective a: Defining the derivative

Recall our discussion on finding the average rate of change between two points on a curve. We drew the secant line and calculated the slope of that line. We then modified the formula for the slope to get the difference quotient:



The slope of the secant line is $m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$

(The symbol delta, Δ, means "change in"). Let $h = x_2 - x_1$ and let $x = x_1$. Then $x_2 = x_1 + h = x + h$. Plugging into m, we get:

$$m = \frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{f(x+h) - f(x)}{h}.$$

The quotient, $\frac{f(x+h) - f(x)}{h}$, is called the difference quotient. Keep in mind that this is another formula for the slope of the line between two points on the curve.

Ex. 1 Given that $f(x) = x^2$, find the difference quotient when

a) $x = 2$ and $h = 0.1$.

b) $x = 2$ and $h = 0.01$.

c) $x = 2$ and $h = 0.001$.

d) $x = 2$ and $h \to 0$

Solution:

a) $\frac{f(x+h)-f(x)}{h} = \frac{f(2+0.1)-f(2)}{0.1} = \frac{f(2.1)-f(2)}{0.1} = \frac{(2.1)^2-(2)^2}{0.1}$

$= \frac{4.41-4}{0.1} = \frac{0.41}{0.1} = 4.1$. So, the slope of the line or the average rate of change is 4.1.

b) $\frac{f(x+h)-f(x)}{h} = \frac{f(2+0.01)-f(2)}{0.01} = \frac{f(2.01)-f(2)}{0.01} = \frac{(2.01)^2-(2)^2}{0.01}$

$= \frac{4.0401-4}{0.01} = \frac{0.0401}{0.01} = 4.01$. So, the slope of the line or the average rate of change is 4.01.

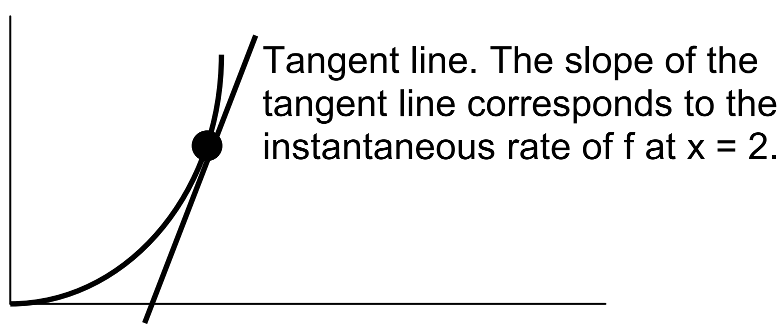
c) $\frac{f(x+h)-f(x)}{h} = \frac{f(2+0.001)-f(2)}{0.001} = \frac{f(2.001)-f(2)}{0.001}$

$= \frac{(2.001)^2-(2)^2}{0.001} = \frac{4.004001-4}{0.001} = \frac{0.004001}{0.001} = 4.001$. So, the slope of the line or the average rate of change is 4.001.

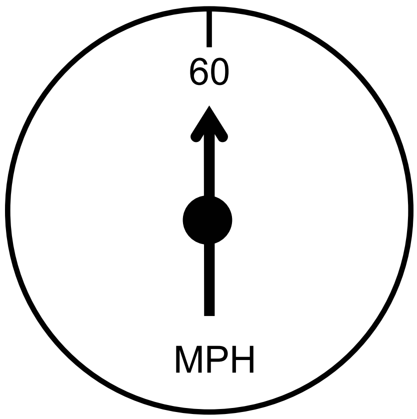
d) $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{f(2+h)-f(2)}{h}$. But $f(2 + h) = (2 + h)^2 = 4 + 4h + h^2$ and $f(2) = (2)^2 = 4$. So,

$\lim_{h\to 0} \frac{f(2+h)-f(2)}{h} = \lim_{h\to 0} \frac{4+4h+h^2-4}{h} = \lim_{h\to 0} \frac{4h+h^2}{h}$

$= \lim_{h\to 0} \frac{h(4+h)}{h} = \lim_{h\to 0} 4 + h = 4 + (0) = 4$. This is the **instantaneous** rate of change of f at x = 2. This is the slope of the **tangent line** at x = 2.



$m = \frac{f(x+h)-f(x)}{h}$ corresponds to the average rate of change. For example, if you traveled 171 miles in 3 hours, your average speed was $\frac{171}{3}$ = 51 mph. This would be your average rate of change. $\lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$ corresponds to the instantaneous rate of change. It is as if you took a picture of your speedometer. The reading on the speedometer tells you how fast you are going at a particular moment. This is your instantaneous rate of change.



We call this instantaneous rate of change **the derivative of the function f with respect to x.** We denote this as $f'(x)$ ("f prime of x"), y' ("y prime"), $\frac{dy}{dx}$ ("dee y dee x"), or $\frac{df}{dx}$ ("dee f dee x"). Hence, we can make the following definition:

Definition: The derivative of the function f with respect to x is:

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx}f(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}.$$

Objective b: Applying the definition of the derivative.

Ex. 2 For the following function, compute $f'(x)$

$f(x) = x^2$

<u>Solution:</u>

$f(x + h) = (x + h)^2 = x^2 + 2xh + h^2$ and $f(x) = x^2$.

Thus, $f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$

$= \lim_{h \to 0} \frac{x^2+2xh+h^2-x^2}{h} = \lim_{h \to 0} \frac{2xh+h^2}{h} = \lim_{h \to 0} \frac{h(2x+h)}{h}$

$= \lim_{h \to 0} 2x + h = 2x + 0 = 2x$. Thus. $f'(x) = 2x$.

Ex. 3 For the following function, compute f '(x) and find the equation of the tangent line at the given value of x.

$f(x) = x^3 – 1$; $x = 2$

<u>Solution:</u>

$f(x + h) = (x + h)^3 – 1 = x^3 + 3x^2h + 3xh^2 + h^3 – 1$ and

$f(x) = x^3 – 1$. Thus, $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$

$= \lim_{h\to 0} \frac{x^3+3x^2h+3xh^2+h^3-1-(x^3-1)}{h}$

$= \lim_{h\to 0} \frac{x^3+3x^2h+3xh^2+h^3-1-x^3+1}{h} = \lim_{h\to 0} \frac{3x^2h+3xh^2+h^3}{h}$

$= \lim_{h\to 0} \frac{h(3x^2+3xh+h^2)}{h} = \lim_{h\to 0} 3x^2 + 3xh + h^2 = 3x^2 + 0 + 0$

$= 3x^2$. Thus, $f'(x) = 3x^2$.

When $x = 2$, then $m = f'(2) = 3(2)^2 = 12$ and

$f(2) = (2)^3 – 1 = 8 – 1 = 7$.

Using the point-slope formula, we can find the equation of the tangent line:

$y – y_1 = m(x – x_1)$

$y – 7 = 12(x – 2)$

$y – 7 = 12x – 24$

$y = 12x – 17$.

Ex. 4 For the following function, compute f '(p) and find the equation of the tangent line at the given value of p.

$f(p) = \frac{1}{\sqrt{p}}$; $p = 9$ where $p \geq 0$.

<u>Solution:</u>

$f'(p) = \lim_{h\to 0} \frac{f(p+h)-f(p)}{h} = \lim_{h\to 0} \frac{\frac{1}{\sqrt{p+h}} - \frac{1}{\sqrt{p}}}{h}$

$= \lim_{h\to 0} \left[\left(\frac{1}{\sqrt{p+h}} - \frac{1}{\sqrt{p}}\right) \div h\right]$

$= \lim_{h\to 0} \left[\left(\frac{1}{\sqrt{p+h}} \bullet \frac{\sqrt{p}}{\sqrt{p}} - \frac{1}{\sqrt{p}} \bullet \frac{\sqrt{p+h}}{\sqrt{p+h}}\right) \bullet \frac{1}{h}\right]$

$= \lim_{h\to 0} \left[\left(\frac{\sqrt{p} - \sqrt{p+h}}{\sqrt{p}\sqrt{p+h}}\right) \bullet \frac{1}{h}\right]$ Here, we will need to rationalize the numerator by multiplying top and bottom by$(\sqrt{p} + \sqrt{p+h})$.

$$= \lim_{h\to 0}\left[\left(\frac{\sqrt{p}-\sqrt{p+h}}{\sqrt{p}\sqrt{p+h}} \bullet \frac{(\sqrt{p}+\sqrt{p+h})}{(\sqrt{p}+\sqrt{p+h})}\right)\bullet \frac{1}{h}\right]$$

$$= \lim_{h\to 0}\left[\left(\frac{(\sqrt{p})^2-(\sqrt{p+h})^2}{\sqrt{p}\sqrt{p+h}(\sqrt{p}+\sqrt{p+h})}\right)\bullet \frac{1}{h}\right]$$

$$= \lim_{h\to 0}\left[\left(\frac{(p)-(p+h)}{\sqrt{p}\sqrt{p+h}(\sqrt{p}+\sqrt{p+h})}\right)\bullet \frac{1}{h}\right]$$

$$= \lim_{h\to 0}\left[\left(\frac{-h}{\sqrt{p}\sqrt{p+h}(\sqrt{p}+\sqrt{p+h})}\right)\bullet \frac{1}{h}\right]$$

$$= \lim_{h\to 0}\frac{-1}{\sqrt{p}\sqrt{p+h}(\sqrt{p}+\sqrt{p+h})} = \frac{-1}{\sqrt{p}\sqrt{p+(0)}(\sqrt{p}+\sqrt{p+(0)})}$$

$= \frac{-1}{\sqrt{p}\sqrt{p}(\sqrt{p}+\sqrt{p})} = \frac{-1}{p(2\sqrt{p})} = \frac{-1}{2p\sqrt{p}}$. Thus, $f'(p) = \frac{-1}{2p\sqrt{p}}$.

When $p = 9$, then $m = f'(9) = \frac{-1}{2(9)\sqrt{(9)}} = \frac{-1}{54}$ and

$y = f(9) = \frac{1}{\sqrt{(9)}} = \frac{1}{3}$.

Using the point-slope formula, we can find the equation of the tangent line:

$y - y_1 = m(p - p_1)$

$y - \frac{1}{3} = \frac{-1}{54}(p - 9)$

$y - \frac{1}{3} = \frac{-1}{54}p + \frac{1}{6}$

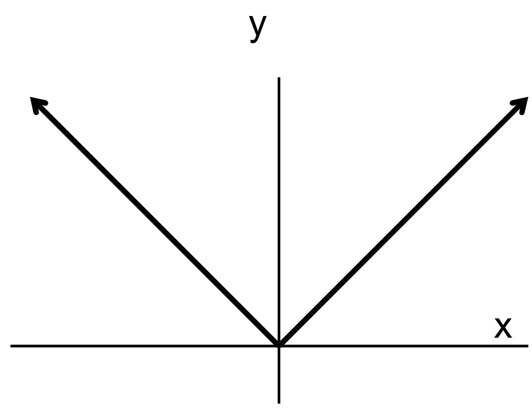
$y = \frac{-1}{54}p + \frac{1}{2}$.

Objective c: Understanding differentiability.

Definition: A function f is differentiable at a point (c, f (c)) if $f'(c)$ exists (in other words, $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ exists at x = c).

Property: If a function is differentiable at x = c, then it is continuous at x = c.
The converse is not true. Consider the following example:
Suppose $f(x) = |x|$. The graph of f looks like:

Ex. 5



Since there are no "breaks" or "holes" in the graph, f is continuous for all real numbers. But, the derivative does not exist at x = 0 since

$$\lim_{h\to 0^+} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^+} \frac{|h|-|0|}{h} = \lim_{h\to 0^+} \frac{|h|}{h} = 1 \text{ and}$$

$$\lim_{h\to 0^-} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^-} \frac{|h|-|0|}{h} = \lim_{h\to 0^-} \frac{|h|}{h} = -1.$$

Typically, a function is not differentiable at x = c if

1) It is not continuous at x = c.

Or

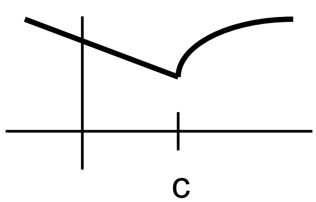
2) It has a "sharp point" or a "sharp corner" at x = c.

Or

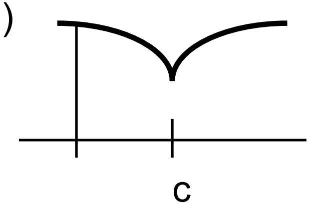
3) It has a vertical tangent line at x = c.

Ex. 6 Consider the following graphs:

a)

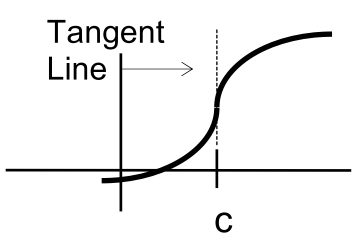


b)

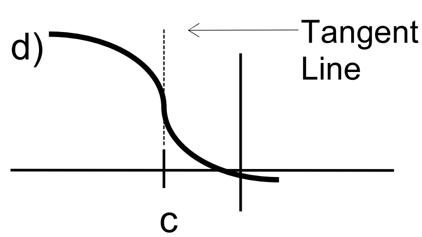


In a) and b), the function is not differentiable at x = c since the graph has a sharp point or corner at x = c.

c)

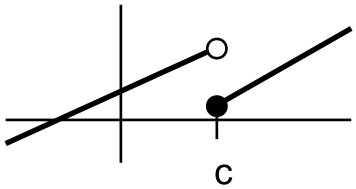


d)

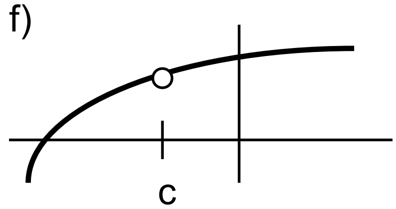


In c) and d), the function is not differentiable at x = c since the tangent line is a vertical line. Thus, the slope of the tangent line is undefined.

e)

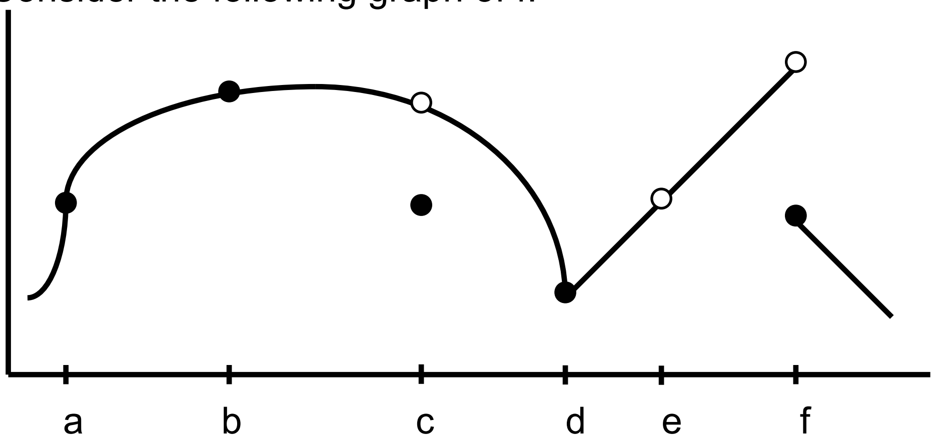


f)



In e) and f), the function is not differentiable at x = c since the function is not continuous at x = c.

Ex. 7 Consider the following graph of f:



I) At which of the labeled points is f continuous?
II) At which of the labeled points is f differentiable?

<u>Solution:</u>

I) The function f is only continuous at the points a, b, & d. It is discontinuous at c since $\lim_{x\to c} f(x) \neq f(c)$. It is discontinuous at e since f(e) is not defined. It is discontinuous at f since $\lim_{x\to f} f(x)$ does not exist.

II) The function f is only differentiable at the point b. It is not differentiable at c, e, and f since it is not continuous at those points (see part I). It is not differentiable at a since the tangent line is vertical and It is not differentiable at d since the function has a sharp corner at d.