# Section 2.5 - Techniques of Differentiation

Objective a: Understanding & using the rules for differentiation.

Our current way of finding the derivative of a function can be long and tedious even for relatively simple functions. In this section, we will develop some techniques that will greatly simplify our work.

For any horizontal line (y = constant), the slope of the line is zero. Recall that the derivative of a function is the slope of the tangent line at a given point. For a horizontal line, the tangent line at any given point will be a horizontal line which has a slope of zero. So, the derivative of constant is zero.

For any linear function, the slope of the tangent line will be equal to the slope of the line. Thus, the derivative of $y = mx + b$ is m. We can then write down some rules for differentiation:

**Rules for differentiation.**
Assume f & g are differentiable functions and m, n, b, & c are constants, then

1) $\frac{d}{dx}[c] = 0.$
2) $\frac{d}{dx}[mx + b] = m.$
3) $\frac{d}{dx}[x^n] = nx^{n-1}$ (power rule).
4) $\frac{d}{dx}[c\bullet f(x)] = c\bullet\frac{d}{dx}[f(x)].$
5) $\frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx}[f(x)] \pm \frac{d}{dx}[g(x)]$

Ex. 1 Differentiate the following:

a) $y = x^3 + 3x^2 - 6x + 5$

Solution:

$y' = \frac{d}{dx}[x^3] + 3\bullet\frac{d}{dx}[x^2] - \frac{d}{dx}[6x] + \frac{d}{dx}[5]$

$y' = 3x^{3-1} + 3\bullet 2x^{2-1} - 6 + 0$

$y' = 3x^2 + 6x - 6.$

b) $f(x) = x^{-4} + x^{3/2}$

Solution:

$f'(x) = \frac{d}{dx}[x^{-4}] + \frac{d}{dx}[x^{3/2}] = -4x^{-4-1} + \frac{3}{2}x^{3/2-1}$

$= -4x^{-5} + \frac{3}{2}x^{1/2} = -\frac{4}{x^5} + \frac{3}{2}\sqrt{x}$.

The properties of exponents that we are using in the last step are:

1) $x^{-n} = \frac{1}{x^n}$

2) $\sqrt[n]{x} = x^{\frac{1}{n}}$

3) $\sqrt[n]{x^m} = x^{\frac{m}{n}}$ or $\frac{1}{\sqrt[n]{x^m}} = x^{-\frac{m}{n}}$

c) $g(x) = \frac{3}{\sqrt{x}} - \frac{5}{x^3} + \frac{7}{\sqrt[3]{x^5}}$

Solution:

Before we can differentiate, we must use our properties of exponents to write each term as a power:

$g(x) = \frac{3}{x^{\frac{1}{2}}} - \frac{5}{x^3} + \frac{7}{x^{\frac{5}{3}}} = 3x^{-1/2} - 5x^{-3} + 7x^{-5/3}$.

Thus, $g'(x) = 3\bullet\frac{d}{dx}[x^{-1/2}] - 5\bullet\frac{d}{dx}[x^{-3}] + 7\bullet\frac{d}{dx}[x^{-5/3}]$

$= 3\bullet(\frac{-1}{2})[x^{-1/2-1}] - 5\bullet(-3)[x^{-3-1}] + 7\bullet(\frac{-5}{3})[x^{-5/3-1}]$

$= \frac{-3}{2}x^{-3/2} + 15x^{-4} + \frac{-35}{3}x^{-8/3}$

$= \frac{-3}{2x^{\frac{3}{2}}} + \frac{15}{x^4} + \frac{-35}{3x^{\frac{8}{3}}} = -\frac{3}{2x\sqrt{x}} + \frac{15}{x^4} - \frac{35}{3x^2\sqrt[3]{x^2}}$.

d) $y = \frac{x^2(x^3 - 8x + 3)}{x^3}$

Solution:

Before we can differentiate, we must simplify the expression:

$y = \frac{x^2(x^3 - 8x + 3)}{x^3} = \frac{x^5 - 8x^3 + 3x^2}{x^3} = x^2 - 8 + 3x^{-1}$

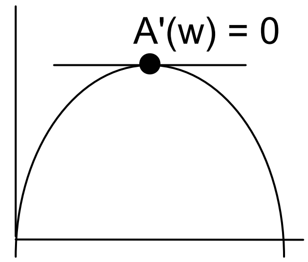
Thus, $y' = \frac{d}{dx}[x^2] - \frac{d}{dx}[8] + 3\bullet\frac{d}{dx}[x^{-1}]$

$= 2x - 0 + 3(-1)x^{-2} = 2x - \frac{3}{x^2}$.

Objective b: Examining horizontal tangents.

Ex. 2 A landscaper wishes to enclose a rectangular garden with 40 feet of fencing. Use calculus to find the largest area that can be enclosed by such a garden.

Solution:
Let L = the length of the garden
and w = the width of the garden
Since we have forty feet of fencing, the perimeter of the garden has to be 40 ft. Thus, 2L + 2w = 40. Solving for L yields: L = 20 – w. The area of a rectangle is A = L•w.
Substituting in for L yields:
$A(w) = L \cdot w = (20 - w)w = 20w - w^2$.
This is a parabola that points down. The maximum value of the area will occur at the vertex or where the derivative is 0. Computing the derivative, we get:




$$A'(w) = \frac{d}{dw}[20w - w^2] = \frac{d}{dw}[20w] - \frac{d}{dw}[w^2] = 20 - 2w.$$

Setting the derivative equal to zero and solving yields:

$$20 - 2w = 0$$
$$-2w = -20$$
$$w = 10 \text{ feet}$$

Since L = 20 – w, then L = 20 – (10) = 10 feet.
Thus, the dimensions of the garden should be 10 ft by 10 ft to enclose a maximum area of 100 $ft^2$.

In general, for a quadratic equation, we can find the x - coordinate of the vertex of the parabola (the highest or lowest point depending on if it is point up or down) by taking the derivative of the function, setting it equal to zero and solving:

$$y = ax^2 + bx + c$$
$$y' = \frac{d}{dx}[ax^2 + bx + c] = \frac{d}{dx}[ax^2] + \frac{d}{dx}[bx] + \frac{d}{dx}[c] = 2ax + b.$$

So, $y' = 2ax + b = 0$

$$2ax = -b$$
$$x = \frac{-b}{2a}$$

. This should look very familiar to you. Now you know where the formula comes from!

**For each function, find the points on the graph at which the tangent line is horizontal:**

Ex. 3a $y = x^3 - 12x + 13$
Ex. 3b $g(x) = 10x^6 + 8x^5 - 5x^4 + 3$
Ex. 3c $f(x) = 3x - 4$
Ex. 3d $y = 7$

Solution:
A horizontal line has a slope equal to 0. Since the derivative is the slope of the tangent line, we are looking for the points where the derivative is equal to zero.

a) $\frac{dy}{dx} = \frac{d}{dx}(x^3) - \frac{d}{dx}(12x) + \frac{d}{dx}(13) = 3x^2 - 12 + 0$
$= 3x^2 - 12.$
Setting $\frac{dy}{dx} = 0$ and solving yields:
$3x^2 - 12 = 0$
$3(x^2 - 4) = 0$
$3(x - 2)(x + 2) = 0$
$x = 2$ or $x = -2$
$y\Big|_{x=2} = (2)^3 - 12(2) + 13 = 8 - 24 + 13 = -3$
$y\Big|_{x=-2} = (-2)^3 - 12(-2) + 13$
$= -8 + 24 + 13 = 29$
Thus, the tangent line is horizontal at the points (2, – 3) and (– 2, 29).

b) $g'(x) = \frac{d}{dx}(10x^6) + \frac{d}{dx}(8x^5) - \frac{d}{dx}(5x^4) + \frac{d}{dx}(3)$
$= 60x^5 + 40x^4 - 20x^3$
Setting $g'(x) = 0$ and solving yields:
$60x^5 + 40x^4 - 20x^3 = 0$
$20x^3(3x^2 + 2x - 1) = 0$
$20x^3(3x - 1)(x + 1) = 0$
$x = 0$, $x = \frac{1}{3}$, or $x = -1$
$g(0) = 10(0)^6 + 8(0)^5 - 5(0)^4 + 3 = 3$
$g(\frac{1}{3}) = 10(\frac{1}{3})^6 + 8(\frac{1}{3})^5 - 5(\frac{1}{3})^4 + 3$
$= \frac{10}{729} + \frac{8}{243} - \frac{5}{81} + 3 = \frac{2176}{729}$

$g(-1) = 10(-1)^6 + 8(-1)^5 - 5(-1)^4 + 3$
$= 10 - 8 - 5 + 3 = 0$

Thus, the tangent line is horizontal at the points $(0, 3)$, $(\frac{1}{3}, \frac{2176}{729})$, and $(-1, 0)$.

c) $f'(x) = \frac{d}{dx}(3x) - \frac{d}{dx}(4) = 3$

Setting $f'(x) = 0$ and solving yields:

$3 = 0$

No Solution

Thus, there are no points on the function f that have horizontal tangents.

d) $\frac{dy}{dx} = \frac{d}{dx}(7) = 0$

Setting $\frac{dy}{dx} = 0$ and solving yields:

$0 = 0$

All real numbers.

Thus, every point on the line $y = 7$ has a horizontal tangent.