# Section 2.6 - Instantaneous Rate of Change

Objective a: Understanding applications of instantaneous rates of change.

Recall that the instantaneous rate of change of a function with respect to x is given by the derivative:

$$f'(x) = \frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

In science, the function s(t) is often used to describe the **position** (think distance) of an object moving along a straight line as a function of time t. The function v(t) is used for the **velocity** of the object (think speed) and the function a(t) is used for the **acceleration** (think how fast an object is speeding up or slowing down). The velocity function is the rate of change of position with respect to time, so v(t) = s '(t). Whereas the acceleration function is the rate of change of "speed" with respect to time, so a(t) = v '(t).

Ex. 1 Given the position function $s(t) = 7t^2 - 28t + 5$, $t \geq 0$ and t is measured in seconds, find:

a) The velocity and acceleration function.

b) Find all the times the object is stationary.

Solution:

a) $v(t) = s'(t) = \frac{d}{dt}[7t^2 - 28t + 5] = 14t - 28.$

$a(t) = v'(t) = \frac{d}{dt}[14t - 28] = 14.$

b) An object is stationary when the velocity is equal to zero. Thus, setting v(t) = 0 and solving yields:

$14t - 28 = 0$

$14t = 28$

$t = 2$

Hence, at t = 2 seconds, the object is stationary.

Ex. 2 Given the position function $s(t) = t^4 - 6t^2 + 8t$, $t \geq 0$ and t is measured in seconds, find:

a) The velocity and acceleration function.

b) Find all the times the object is stationary.

<u>Solution:</u>

a) $v(t) = s'(t) = \frac{d}{dt}[t^4 - 6t^2 + 8t] = 4t^3 - 12t + 8.$

$a(t) = v'(t) = \frac{d}{dt}[4t^3 - 12t + 8] = 12t^2 - 12.$

b) An object is stationary when the velocity is equal to zero. Thus, setting v(t) = 0 and solving yields:

$4t^3 - 12t + 8 = 0$

$4(t^3 - 3t + 2) = 0$

The possible rational zeros are ± 1 and ± 2. Using synthetic division, we find:

| | | | | |
|---|---|---|---|---|
| 1 | 1 | 0 | – 3 | 2 |
| | | 1 | 1 | – 2 |
| | 1 | 1 | – 2 | 0 |

So, t – 1 is a factor of $4(t^3 - 3t + 2)$ and our problem becomes:

$4(t - 1)(t^2 + t - 2) = 0$

$4(t - 1)(t + 2)(t - 1) = 0$

t = 1 or t = – 2. But t = – 2 is not in the domain of the function since t ≥ 0. Thus, t = 1 is the only solution. Hence, at t = 1 second, the object is stationary.

Ex. 3 The circular area A, in $m^2$, of an oil spill is given by: $A = \pi r^2$, where r is the radius of the oil spill in meters.

a) Find the rate of change of the area with respect to the radius.

b) Find the rate of change of the area at r = 7 m.

c) What does the answer in part b mean?

<u>Solution:</u>

a) $\frac{dA}{dr} = \frac{d}{dr}(\pi r^2) = 2\pi r$

b) $\left.\frac{dA}{dr}\right|_{r=7} = 2\pi(7) = 14\pi\ m^2/m = 14\pi\ m$

c) The rate of change of the area of the oil spill is $14\pi\ m^2$ per 1 m of change in the radius.

Ex. 4 The number of new homes sold h (in thousands of homes) in the United States from July, 2004 to July, 2005 can be modeled by:

$h(t) = \frac{2}{3}t^3 - 13.55245t^2 + 92.016127t + 1036.51049$

where t is the time in months after June of 2004.

a) Find the rate of change of the number of new homes sold with respect to time.

b) Find the rate of change of the number of new homes sold in December, 2004 (t = 6).

c) What does the answer in part b mean?

Solution:

a) $h'(t) = \frac{d}{dt}\left(\frac{2}{3}t^3\right) - \frac{d}{dt}(13.55245t^2) + \frac{d}{dt}(92.016127t) + \frac{d}{dt}(1036.51049)$

$= 2t^2 - 27.1049t + 92.016127$

b) $h'(6) = 2(6)^2 - 27.1049(6) + 92.016127$

$= 72 - 162.6294 + 92.016127 = 1.386727$ thousand per month.

c) Thus, the number of new homes sold was increasing at a rate of ≈ 1387 homes per month in December, 2004

In business and economics, we are interested in examining the cost, the revenue and the profit for a particular commodity. The total cost function C(x) is the total cost of producing x units of a product. The total revenue function R(x) is revenue received from selling x units of a product. The total profit function P(x) = R(x) – C(x) is the total profit from producing and selling x units of a product. The marginal cost, the marginal revenue, and the marginal profit measures the rate of the respective catagories with respect to the number of units produced. In other words:

The **Marginal Cost**, C'(x), is the rate of change of the total cost with respect to the number of x units produced ("the cost to produce one more unit or the $(x + 1)^{st}$ unit")

The **Marginal Revenue**, R'(x), is the rate of change of the total revenue with respect to the number of x units sold ("the amount of revenue received from selling one more or the $(x + 1)^{st}$ unit").

The **Marginal Profit**, P'(x), is the rate of change of the total profit with respect to the number of x units produced and sold (“the amount of profit received from producing and selling one more or the $(x + 1)^{st}$ unit").
Since P(x) = R(x) – C(x), then P'(x) = R'(x) – C'(x).

Ex. 5 Let $C(x) = \frac{3}{5}x^2 + 5x + 102$ be the total cost function for producing x units of a particular commodity and $p(x) = -2x^2 + 3x + 62$ be the price at which all x units are sold.

Solution:

a) Find the marginal revenue and the marginal cost.

b) Use the marginal cost to estimate the cost of producing the $4^{th}$ unit (x = 3).

c) Use the marginal revenue to estimate the revenue derived from the sale of the $4^{th}$ unit (x = 3).

d) Find the marginal profit.

e) Use the marginal profit to estimate the profit derived from the production and sale of the $4^{th}$ unit (x = 3).

f) If the average cost is $\frac{C(x)}{x}$, compute the average cost and the marginal average cost.

Solution:

a) The total revenue is x•p(x), so
$R(x) = x(-2x^2 + 3x + 62) = -2x^3 + 3x^2 + 62x.$
Computing the derivative, we get:
$R'(x) = \frac{d}{dx}[-2x^3 + 3x^2 + 62x] = -6x^2 + 6x + 62.$
Since $C(x) = \frac{3}{5}x^2 + 5x + 102$, then
$C'(x) = \frac{d}{dx}[\frac{3}{5}x^2 + 5x + 102] = \frac{6}{5}x + 5.$

b) To find the cost of producing the $4^{th}$ unit ($(x + 1)^{st}$ unit), we need to evaluate C'(x) at x = 3 units:
$C'(3) = \frac{6}{5}(3) + 5 = 3.6 + 5 = \$8.60$
The cost to produce the $4^{th}$ unit is about \$8.60.

c) To find the revenue of selling the 4th unit (($x_o$ + 1)st unit), we need to evaluate R'(x) at x = 3 units:
$R'(3) = -6(3)^2 + 6(3) + 62 = -54 + 18 + 62 = 26$.
The revenue from selling the 4th unit is about $26.

d) Since P'(x) = R'(x) – C'(x), then P'(x) =
$-6x^2 + 6x + 62 - (\frac{6}{5}x + 5) = -6x^2 + 4.8x + 57$.

e) To find the profit of producing and selling the 4th unit ((x + 1)st unit), we need to evaluate P'(x) at x = 3 units: $P'(3) = -6(3)^2 + 4.8(3) + 57$
$= -54 + 14.4 + 57 = 17.4$.
The profit from producing and selling the 4th unit is about $17.40.

f) We first find the average cost function:

$$A = \frac{C(x)}{x} = \frac{\frac{3}{5}x^2 + 5x + 102}{x} = \frac{3}{5}x + 5 + \frac{102}{x}$$

Computing the derivative, we get:

$$A'(x) = \frac{d}{dx}\left[\frac{3}{5}x + 5 + \frac{102}{x}\right] = \frac{d}{dx}\left[\frac{3}{5}x + 5 + 102x^{-1}\right]$$

$$= \frac{3}{5} - 102x^{-2} = \frac{3}{5} - \frac{102}{x^2}.$$