Section 2.7 – The Product and Quotient Rules

In this section, we turn our attention to differentiating the product and quotient of two functions. Unfortunately, we cannot simply differentiate each function like we did with the sum and difference of functions.

Product Rule: If f(x) and g(x) are differentiable at x, then

$$\frac{d}{dx}[f(x)\bullet g(x)] = f(x)\bullet\frac{d}{dx}[g(x)] + g(x)\bullet\frac{d}{dx}[f(x)]$$

As a memory aid, we typically think of the product rule as:

"First • The Derivative of the Second **+** Second • The Derivative of the First"

Differentiate the following functions two different ways:

a) Use the product rule and simplify.
b) Expand and then differentiate.

Ex. 1 $h(x) = 15x^3(6x^2)$

Solution:

a) $h'(x) = (15x^3)\bullet\frac{d}{dx}[6x^2] + 6x^2\bullet\frac{d}{dx}[15x^3]$
$= (15x^3)\bullet[12x] + 6x^2\bullet[45x^2]$
$= 180x^4 + 270x^4 = 450x^4.$

b) $h(x) = 15x^3(6x^2) = 90x^5$
$h'(x) = \frac{d}{dx}[90x^5] = 450x^4.$

Notice that the results are the same.

Ex. 2 $y = (2x^2 + 3x)(x + \sqrt{x})$

Solution:

a) $y' = (2x^2 + 3x)\bullet\frac{d}{dx}[x + \sqrt{x}] + (x + \sqrt{x})\bullet\frac{d}{dx}[2x^2 + 3x]$
$= (2x^2 + 3x)\bullet\frac{d}{dx}[x + x^{1/2}] + (x + x^{1/2})\bullet\frac{d}{dx}[2x^2 + 3x]$
$= (2x^2 + 3x)\bullet[1 + \frac{1}{2}x^{-1/2}] + (x + x^{1/2})\bullet[4x + 3]$
$= (2x^2 + 3x)\bullet[1 + \frac{1}{2x^{1/2}}] + (x + x^{1/2})\bullet[4x + 3]$ (**FOIL**)

$= 2x^2 + \frac{2x^2}{2x^{1/2}} + 3x + \frac{3x}{2x^{1/2}} + 4x^2 + 3x + 4x^{3/2} + 3x^{1/2}$

$= 2x^2 + \underline{x^{3/2}} + 3x + \mathbf{1.5x^{1/2}} + 4x^2 + 3x + \underline{4x^{3/2}} + \mathbf{3x^{1/2}}$

$= 6x^2 + 5x^{3/2} + 6x + 4.5x^{1/2}$

$= 6x^2 + 5x\sqrt{x} + 6x + 4.5\sqrt{x}$.

b) $y = (2x^2 + 3x)(x + \sqrt{x}) = (2x^2 + 3x)(x + x^{1/2})$

$= 2x^3 + 2x^{5/2} + 3x^2 + 3x^{3/2}$

$y' = \frac{d}{dx}[2x^3 + 2x^{5/2} + 3x^2 + 3x^{3/2}]$

$= 6x^2 + 2(5/2)x^{3/2} + 6x + 3(3/2)x^{1/2}$

$= 6x^2 + 5x^{3/2} + 6x + 4.5x^{1/2}$

$= 6x^2 + 5x\sqrt{x} + 6x + 4.5\sqrt{x}$.

Again, notice the results are the same.

It is important to know and to understand how to use the product rule. Later in this chapter, we will be differentiating the product of two functions where we can not easily simplify first. Now, let's examine the quotient rule:

Quotient Rule: If n(x) and d(x) are differentiable at x and $d(x) \neq 0$ at x, then

$$\frac{d}{dx}\left[\frac{n(x)}{d(x)}\right] = \frac{d(x)\bullet\frac{d}{dx}[n(x)] - n(x)\bullet\frac{d}{dx}[d(x)]}{[d(x)]^2}$$

Here, **<u>n</u>(x)** is the function in the **<u>n</u>umerator** while **d(x)** is the function in the **denominator**.

Differentiate the following functions:

Ex. 3 $f(x) = \frac{x^2 + 11x - 5}{6x + 1}$.

<u>Solution:</u>

Since $n(x) = x^2 + 11x - 5$ and $d(x) = 6x + 1$, then

$n'(x) = \frac{d}{dx}[x^2 + 11x - 5] = 2x + 11$ & $d'(x) = \frac{d}{dx}[6x + 1] = 6$.

Thus, $\frac{d}{dx}\left[\frac{x^2 + 11x - 5}{6x + 1}\right] = \frac{d(x)\bullet\frac{d}{dx}[n(x)] - n(x)\bullet\frac{d}{dx}[d(x)]}{[d(x)]^2}$

$$= \frac{(6x+1)[2x+11]-(x^2+11x-5)[6]}{[6x+1]^2} = \frac{12x^2+68x+11-6x^2-66x+30}{[6x+1]^2}$$

$$= \frac{6x^2+2x+41}{[6x+1]^2}.$$

Note, if you divide first in and then differentiate, you get:

$$\begin{array}{r|l} & \frac{1}{6}x + \frac{65}{36} - \frac{\frac{245}{36}}{6x+1} \\ \hline 6x+1 & x^2 + 11x - 5 \\ & -x^2 - \frac{1}{6}x \\ \hline & \frac{65}{6}x - 5 \\ & -\frac{65}{6}x - \frac{65}{36} \\ \hline & -\frac{245}{36} \end{array}$$

Thus, $f'(x) = \frac{d}{dx}[\frac{1}{6}x + \frac{65}{36} - \frac{\frac{245}{36}}{6x+1}] = \frac{1}{6} - \frac{d}{dx}[\frac{\frac{245}{36}}{6x+1}]$

$$= \frac{1}{6} - \frac{(6x+1)\bullet\frac{d}{dx}\left[\frac{245}{36}\right]-(\frac{245}{36})\bullet\frac{d}{dx}[6x+1]}{\left[6x+1\right]^2}$$

$$= \frac{1}{6} - \frac{(6x+1)\bullet\left[0\right]-(\frac{245}{36})\bullet[6]}{\left[6x+1\right]^2} = \frac{1}{6} - \frac{-\frac{245}{6}}{\left[6x+1\right]^2}$$

$$= \frac{1}{6} + \frac{245}{6\left[6x+1\right]^2} = \frac{1}{6}\bullet\frac{\left[6x+1\right]^2}{\left[6x+1\right]^2} + \frac{245}{6\left[6x+1\right]^2}$$

$$= \frac{36x^2+12x+1}{6\left[6x+1\right]^2} + \frac{245}{6\left[6x+1\right]^2} = \frac{36x^2+12x+246}{6\left[6x+1\right]^2} = \frac{6x^2+2x+41}{\left[6x+1\right]^2}.$$

Although you get the same result, the second way is a lot more difficult.

Ex. 4 $r(x) = \frac{2x^2+5x-1}{3x-2}$

Solution:

$$r'(x) = \frac{(3x-2)\bullet\frac{d}{dx}\left[2x^2+5x-1\right]-(2x^2+5x-1)\bullet\frac{d}{dx}[3x-2]}{[3x-2]^2}$$

$$= \frac{(3x-2)[4x+5]-(2x^2+5x-1)[3]}{[3x-2]^2} = \frac{12x^2+7x-10-6x^2-15x+3}{[3x-2]^2}$$

$$= \frac{6x^2-8x-7}{[3x-2]^2}.$$

Ex. 5 $p(x) = \frac{1}{x-5}$

Solution:

$$p'(x) = \frac{(x-5)\bullet\frac{d}{dx}[1]-(1)\bullet\frac{d}{dx}[x-5]}{[x-5]^2} = \frac{(x-5)[0]-(1)[1]}{[x-5]^2} = \frac{-1}{[x-5]^2}.$$

Ex. 6 $h(x) = \frac{(x^2+x+1)(x-4)}{2x^2-1}$

Solution:

First simplify the numerator:

$$h(x) = \frac{(x^2+x+1)(x-4)}{2x^2-1} = \frac{x^3+x^2+x-4x^2-4x-4}{2x^2-1}$$

$$= \frac{x^3-3x^2-3x-4}{2x^2-1}.$$

Now, differentiate using the quotient rule.

$$h'(x) = \frac{(2x^2-1)\bullet\frac{d}{dx}[x^3-3x^2-3x-4]-(x^3-3x^2-3x-4)\bullet\frac{d}{dx}[2x^2-1]}{\left[2x^2-1\right]^2}$$

$$= \frac{(2x^2-1)\bullet\frac{d}{dx}[x^3-3x^2-3x-4]-(x^3-3x^2-3x-4)\bullet\frac{d}{dx}[2x^2-1]}{\left[2x^2-1\right]^2}$$

$$= \frac{(2x^2-1)[3x^2-6x-3]-(x^3-3x^2-3x-4)[4x]}{\left[2x^2-1\right]^2}$$

$$= \frac{2x^4+3x^2+22x+3}{\left[2x^2-1\right]^2}.$$

Ex. 7 Suppose on a certain city, it is known that when x million dollars is spent on controlling pollution, the percentage of pollution removed is given by $P(x) = \frac{100\sqrt{x}}{0.05x^2 + 25.35}$, $x \geq 0$. Determine the expenditure that results in the largest percentage of pollution removal. What is this maximum percent?

Solution:

We begin by differentiating P(x) using the quotient rule:

$$P'(x) = \frac{(0.05x^2 + 25.35)\bullet\frac{d}{dx}\left[100x^{1/2}\right] - (100x^{1/2})\frac{d}{dx}\left[0.05x^2 + 25.35\right]}{\left[0.05x^2 + 25.35\right]^2}$$

$$= \frac{(0.05x^2 + 25.35)[50x^{-1/2}] - (100x^{1/2})[0.1x]}{\left[0.05x^2 + 25.35\right]^2}$$

$$= \frac{2.5x^{3/2} + 1267.5x^{-1/2} - 10x^{3/2}}{\left[0.05x^2 + 25.35\right]^2}$$

$$= \frac{-7.5x^{3/2} + 1267.5x^{-1/2}}{\left[0.05x^2 + 25.35\right]^2}.$$ Recall that a maximum and a minimum value can occur when the slope of the tangent line is 0. Setting the derivative equal to zero and solving yields:

$$\frac{-7.5x^{3/2} + 1267.5x^{-1/2}}{\left[0.05x^2 + 25.35\right]^2} = 0$$

Since $0.05x^2 + 25.35 \neq 0$, then we can multiply both sides by that expression:

$$\mathbf{(0.05x^2 + 25.35)^2}\bullet\frac{-7.5x^{3/2} + 1267.5x^{-1/2}}{\left[0.05x^2 + 25.35\right]^2} = \mathbf{(0.05x^2 + 25.35)^2}\bullet 0$$

$$-7.5x^{3/2} + 1267.5x^{-1/2} = 0$$

Now, multiply both sides by $x^{1/2}$.

$$-7.5x^{3/2}\bullet \mathbf{x^{1/2}} + 1267.5x^{-1/2}\bullet \mathbf{x^{1/2}} = 0\bullet \mathbf{x^{1/2}}$$

$$-7.5x^2 + 1267.5 = 0$$

$$-7.5(x^2 - 169) = 0$$

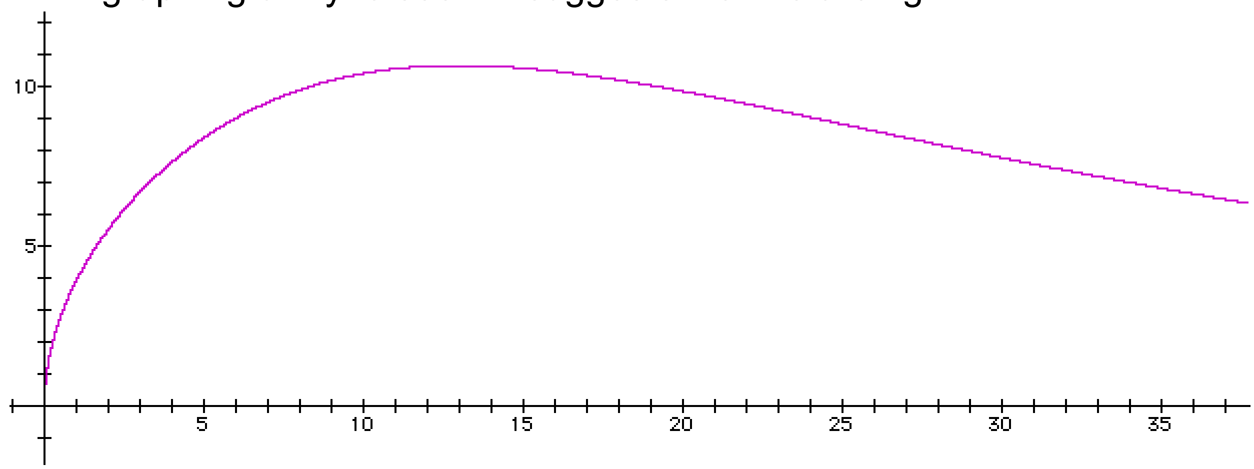
$$-7.5(x - 13)(x + 13) = 0$$

$$x = 13 \text{ or } x = -13$$

But, $x = -13$ is not in the domain of P, so $x = 13$ is the only solution.

Thus, $P(13) = \frac{100\sqrt{(13)}}{0.05(13)^2 + 25.35} \approx \frac{360.55513}{33.8} \approx 10.667\%$

Next, we need to verify that this the maximum value. Since we have not discuss how to do this with calculus (chapter 3), we cannot be 100% sure if we have the maximum. We can look at the graph of the function on a graphing utility to see if it suggests that we are right:



The graph does seem to suggest we found the maximum value. Thus, a maximum of 10.667% of the population is removed when $13 million is spent.

Sometimes the demand function D might be given as a function of the price instead of a function of the number of units sold. Thus, p would be the price per unit and D(p) would be the number of units sold. Likewise, the revenue function can be given as a function of the price since

R(p) = (price per unit)•(number of units sold) = p•D(p)

Ex. 8 Given the demand function $x = D(p) = 200 - \sqrt{p}$, find a) the total revenue R(p) and b) the marginal revenue R'(p).

Solution:

a) $R(p) = p \cdot D(p) = p \cdot (200 - \sqrt{p}) = 200p - p\sqrt{p}$ or $200p - p^{3/2}$. Thus, $R(p) = 200p - p\sqrt{p}$.

b) $R'(p) = \frac{d}{dp}(200p - p^{3/2}) = 200 - \frac{3}{2}p^{1/2} = 200 - \frac{3}{2}\sqrt{p}$.

Thus, $R'(p) = 200 - \frac{3}{2}\sqrt{p}$.