Section 2.8 – Chain Rule

How do we differentiate $y = (5x + 1)^3$? Our first instinct is to apply the power rule:

$y' = \frac{d}{dx}[(5x + 1)^3] = 3(5x + 1)^2$? So, our guess would be:

$3(5x + 1)^2 = 3(25x^2 + 10x + 1) = 75x^2 + 30x + 3.$

We can check this answer by first expanding $(5x + 1)^3$ and then differentiating:

$y = (5x + 1)^3 = (5x + 1)^2(5x + 1) = (25x^2 + 10x + 1)(5x + 1)$

$= 125x^3 + 50x^2 + 5x + 25x^2 + 10x + 1$

$= 125x^3 + 75x^2 + 15x + 1.$

Computing the derivative, we get:

$y' = \frac{d}{dx}[125x^3 + 75x^2 + 15x + 1] = 375x^2 + 150x + 15$. This is not the same as our guess. But notice if we factor out a common factor of 5, we get:

$375x^2 + 150x + 15 = 5(75x^2 + 30x + 3).$

The polynomial inside the parentheses is our guess. We seem only to be missing a factor of five in our initial guess. If we examine 5x + 1, its derivative is 5, the missing factor. Thus if $y = (5x + 1)^3$, then $y' = 3(5x + 1)^2 \bullet \frac{d}{dx}[(5x + 1)] = 3(5x + 1)^2 \bullet (5)$ or $15(5x + 1)^2$. To differentiate a **composition of functions**, we differentiate the outside function and then multiply it by the derivative of the inside function.

Chain Rule: Let O and I be two differentiable functions,
If $h(x) = O[I(x)]$, then $h'(x) = O'[I(x)] \bullet I'(x)$

Ex. 1 Differentiate the following:

a) $y = (x^2 - 3)^{15}$

b) $f(x) = \sqrt{5x^6 - 12}$

c) $g(x) = \frac{(1-2x)^5}{(3x+1)^6}$

Solution:

a) $y' = \frac{d}{dx}[(x^2 - 3)^{15}] = 15(x^2 - 3)^{14} \bullet \frac{d}{dx}[x^2 - 3]$

$= 15(x^2 - 3)^{14} \bullet 2x = 30x(x^2 - 3)^{14}.$

b) $y = \sqrt{5x^6 - 12} = (5x^6 - 12)^{1/2}$

$y' = \frac{d}{dx}[(5x^6 - 12)^{1/2}] = \frac{1}{2}(5x^6 - 12)^{-1/2} \bullet \frac{d}{dx}(5x^6 - 12)$

$= \dfrac{1}{2(5x^6 - 12)^{\frac{1}{2}}} \bullet 30x^5 = \dfrac{15x^5}{\sqrt{5x^6 - 12}}$.

c) Since $g(x) = \dfrac{(1-2x)^5}{(3x+1)^6}$, we will need to use the quotient rule as well as the chain rule:

$$g'(x) = \frac{(3x+1)^6 \bullet \frac{d}{dx}[(1-2x)^5] - (1-2x)^5 \bullet \frac{d}{dx}[(3x+1)^6]}{\left[(3x+1)^6\right]^2}$$

$$= \frac{(3x+1)^6 \bullet 5(1-2x)^4 \frac{d}{dx}[(1-2x)] - (1-2x)^5 \bullet 6(3x+1)^5 \frac{d}{dx}[(3x+1)]}{(3x+1)^{12}}$$

$$= \frac{(3x+1)^6 \bullet 5(1-2x)^4 [-2] - (1-2x)^5 \bullet 6(3x+1)^5 [3]}{(3x+1)^{12}}$$

$$= \frac{-10(3x+1)^6 (1-2x)^4 - 18(1-2x)^5 (3x+1)^5}{(3x+1)^{12}}$$

Factor out a common factor of $-2(3x+1)^5(1-2x)^4$ out of each part of the numerator.

$$= \frac{-2(3x+1)^5 (1-2x)^4 [5(3x+1) + 9(1-2x)]}{(3x+1)^{12}}$$

Five factors of (3x + 1) reduce. Distribute and combine like terms.

$$= \frac{-2(1-2x)^4 [15x + 5 + 9 - 18x]}{(3x+1)^7}$$

$$= \frac{-2(1-2x)^4 (-3x+14)}{(3x+1)^7} \text{ or } \frac{2(1-2x)^4 (3x-14)}{(3x+1)^7}.$$

In this last example, we cannot avoid using the quotient rule in order to differentiate. However, there are some problems where we can use the chain rule to avoid using the quotient rule. It typically involves using our properties of exponents to first rewrite the function as a power and then applying the chain rule. Consider part a of this next example:

Ex. 2 Differentiate:

a) $g(x) = \frac{1}{(7x-3)^6}$

b) $h(x) = \sqrt[3]{x^2 - 7x + 3}$

Solution:

a) First rewrite g(x) as $g(x) = (7x - 3)^{-6}$. Now use the chain rule:

$$g'(x) = \frac{d}{dx}[(7x-3)^{-6}]$$
$$= -6(7x-3)^{-6-1} \bullet \frac{d}{dx}[(7x-3)]$$
$$= -6(7x-3)^{-7} \bullet [7] = -42(7x-3)^{-7}$$
$$= -\frac{42}{(7x-3)^7}.$$

b) First rewrite h(x) as $h(x) = (x^2 - 7x + 3)^{1/3}$. Now use the chain rule:

$$h'(x) = \frac{d}{dx}[(x^2 - 7x + 3)^{1/3}]$$
$$= \frac{1}{3}(x^2 - 7x + 3)^{1/3 - 1} \bullet \frac{d}{dx}[x^2 - 7x + 3]$$
$$= \frac{1}{3}(x^2 - 7x + 3)^{-2/3} \bullet [2x - 7]$$
$$= \frac{1(2x-7)}{3(x^2-7x+3)^{\frac{2}{3}}} = \frac{2x-7}{3\sqrt[3]{(x^2-7x+3)^2}}$$

There is an alternate of the chain rule. Consider the function $y = f[g(x)]$. Let $y = f[u]$ and $u = g(x)$. Then $\frac{dy}{du} = f'[u]$ and $\frac{du}{dx} = g'(x)$. Since $y = f[g(x)]$, then $\frac{dy}{dx} = f'[g(x)] \bullet g'(x)$. But $u = g(x)$, so $\frac{dy}{dx} = f'[u] \bullet g'(x)$. Since $\frac{dy}{du} = f'[u]$ and $\frac{du}{dx} = g'(x)$, then $\frac{dy}{dx} = \frac{dy}{du} \bullet \frac{du}{dx}$.

Chain Rule 2: Let u be a differentiable function of x and y be a differentiable function of u. Then $\frac{dy}{dx} = \frac{dy}{du} \bullet \frac{du}{dx}$.

If you pretend that $\frac{dy}{du}$ and $\frac{du}{dx}$ are fractions, the du's “divide out” leaving $\frac{dy}{dx}$.

Ex. 3 Find the derivative of y with respect to x in the following:

a) $y = \frac{1}{u}$ and $u = 3x^2 + 5$.

b) $y = \frac{1}{u+1}$ and $u = x^3 - 2x + 5$.

<u>Solution:</u>

a) We first compute $\frac{dy}{du}$ and $\frac{du}{dx}$:

$\frac{dy}{du} = \frac{d}{du}[\frac{1}{u}] = \frac{d}{du}[u^{-1}] = -1u^{-2} = \frac{-1}{u^2}$ and

$\frac{du}{dx} = \frac{d}{dx}[3x^2 + 5] = 6x$.

Thus, $\frac{dy}{dx} = \frac{dy}{du} \bullet \frac{du}{dx} = \frac{-1}{u^2} \bullet 6x = \frac{-6x}{u^2}$. But $u = 3x^2 + 5$.

So, $\frac{dy}{dx} = \frac{-6x}{u^2} = -\frac{6x}{(3x^2+5)^2}$.

b) We first compute $\frac{dy}{du}$ and $\frac{du}{dx}$:

$\frac{dy}{du} = \frac{d}{du}[\frac{1}{u+1}] = \frac{d}{du}[(u + 1)^{-1}] = -(u + 1)^{-2}\frac{d}{du}[u + 1]$

$= \frac{-1}{(u+1)^2}[1] = \frac{-1}{(u+1)^2}$ and

$\frac{du}{dx} = \frac{d}{dx}[x^3 - 2x + 5] = 3x^2 - 2$.

Thus, $\frac{dy}{dx} = \frac{dy}{du} \bullet \frac{du}{dx} = \frac{-1}{(u+1)^2} \bullet (3x^2 - 2) = \frac{-(3x^2-2)}{(u+1)^2}$. But

$u = x^3 - 2x + 5$. So, $\frac{dy}{dx} = \frac{-(3x^2-2)}{(u+1)^2} = \frac{-(3x^2-2)}{(x^3-2x+5+1)^2}$

$= \frac{-3x^2+2}{(x^3-2x+6)^2}$.

Ex. 4 At a certain factory, the total cost of manufacturing q units during the daily production run is $C(q) = 0.6q^2 + q + 600$ dollars. From experience it has been determined that approximately $q(t) = t^2 + 50t$ units are manufactured during the first t hours of production. Compute the rate at which the total manufacturing cost is changing **with respect to time** 1 hour after production begins.

Solution:

We first compute $\frac{dC}{dq}$ and $\frac{dq}{dt}$:

$\frac{dC}{dq} = \frac{d}{dq}[0.6q^2 + q + 600] = 1.2q + 1$ and

$\frac{dq}{dt} = \frac{d}{dt}[t^2 + 50t] = 2t + 50$.

Thus, $\frac{dC}{dt} = \frac{dC}{dq} \bullet \frac{dq}{dt} = (1.2q + 1)(2t + 50)$. Since $q(t) = t^2 + 50t$, then $\frac{dC}{dt} = (1.2[t^2 + 50t] + 1)(2t + 50)$ $= (1.2t^2 + 60t + 1)(2t + 50)$

Evaluating $\frac{dC}{dt}$ at t = 1, we get:

$$\left.\frac{dC}{dt}\right|_{t=1} = (1.2(1)^2 + 60(1) + 1)(2(1) + 50) = (62.2)(52)$$

= \$3,234.40.

After one hour, the total manufacturing cost are increasing at a rate of \$3,234.40 per hour.

Ex. 5 If \$4,000 is invested at an annual rate r (expressed as a decimal) compounded weekly, the total amount accumulated after 10 years is given by the formula:

$A(r) = 4000(1 + \frac{r}{52})^{520}$.

a) Find the rate of change of A with respect to r.

b) Find the rate of change of A when r = 0.08.

Solution:

a) Computing the derivative, we get:

$A'(r) = \frac{d}{dr}[4000(1 + \frac{r}{52})^{520}]$

$= 4000 \bullet 520(1 + \frac{r}{52})^{519} \bullet \frac{d}{dr}[1 + \frac{r}{52}]$

$= 4000 \bullet 520(1 + \frac{r}{52})^{519} \bullet [\frac{1}{52}] = 4000 \bullet 10(1 + \frac{r}{52})^{519}$

$= 40000(1 + \frac{r}{52})^{519}$. Thus, $A'(r) = 40000(1 + \frac{r}{52})^{519}$.

b) $A'(0.08) = 40000(1 + \frac{0.08}{52})^{519} = \$88,830.27$.

The accumulated amount is changing by \$88,830.27 for every unit change in r or the accumulated amount is increasing by \$888.30 per 1% increase in the interest rate when the rate is 8%.