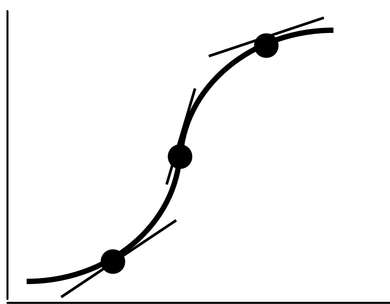
# Section 3.1 - The First Derivative Test

A function f is **increasing** on (a, b) if for all c < d in (a, b), f(c) < f(d).
A function f is **decreasing** on (a, b) if for all c < d in (a, b), f(c) > f(d).

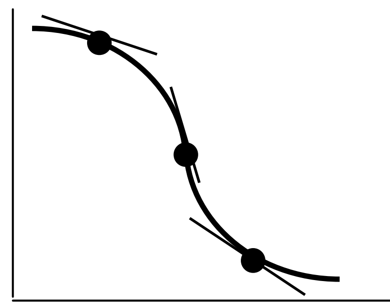
Now, let's look at the relationship between the slopes of the tangent lines and the increasing and decreasing behavior of the function:

Ex. 1 a) Increasing



The slopes of tangent lines are positive.

b) Decreasing



The slopes of tangent lines are negative.

Since the slope of the tangent line is the derivative of the function evaluated at that point, we can use the derivative to determine where the function is increasing and decreasing.

**First Derivative Criteria for where a function is increasing and decreasing:**

1) If f '(x) > 0 for all x in (a, b), then f is increasing on (a, b).
2) If f '(x) < 0 for all x in (a, b), then f is decreasing on (a, b).

Determine the intervals where f is increasing and decreasing:

Ex. 2 $f(x) = \frac{x^2}{x-3}$.

Solution:
We began by finding the domain of the function. The function is a rational expression, it is defined for all real numbers except where the denominator is zero or where x = 3. Thus the domain of f is (– ∞, 3) U (3, ∞).

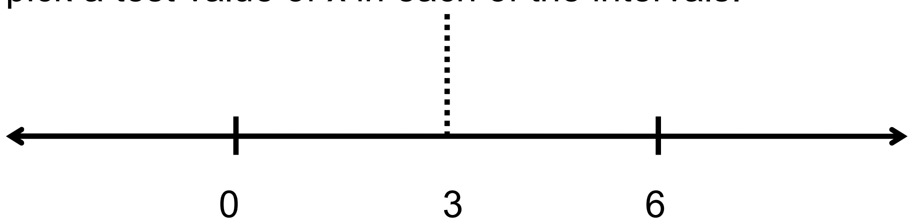
Next, we need to find the derivative:

$$f'(x) = \frac{d}{dx}\left[\frac{x^2}{x-3}\right] = \frac{(x-3)\frac{d}{dx}[x^2] - x^2\frac{d}{dx}[(x-3)]}{(x-3)^2}$$

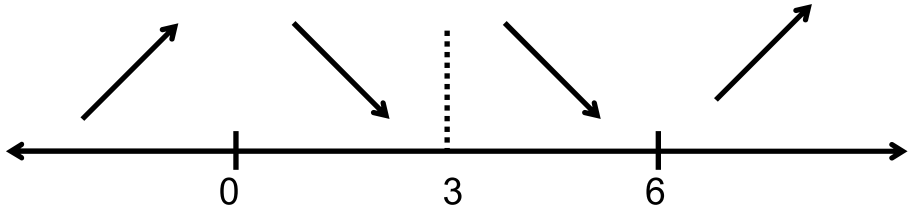
$$= \frac{(x-3)\bullet[2x] - x^2\bullet[1]}{(x-3)^2} = \frac{2x^2 - 6x - x^2}{(x-3)^2} = \frac{x^2 - 6x}{(x-3)^2}$$

$$= \frac{x(x-6)}{(x-3)^2}.$$

The sign of the derivative can "change" when f '(x) is zero or when it is undefined. Thus, we need to split the real numbers at x = 0, x = 3, and x = 6. Hence, we need to find the sign of f '(x) in the intervals (– ∞, 0), (0, 3), (3, 6), and (6, ∞). To do this, we create a sign chart and pick a test value of x in each of the intervals.




| | | | |
|---|---|---|---|
| Pick $x = -1$ | Pick $x = 1$ | Pick $x = 4$ | Pick $x = 7$ |
| $f'(-1) = \frac{(-1)((-1)-6)}{((-1)-3)^2}$ | $f'(1) = \frac{(1)((1)-6)}{((1)-3)^2}$ | $f'(4) = \frac{(4)((4)-6)}{((4)-3)^2}$ | $f'(7) = \frac{(7)((7)-6)}{((7)-3)^2}$ |
| $= \frac{(-1)(-7)}{(-4)^2}$ | $= \frac{(1)(-5)}{(-2)^2}$ | $= \frac{(4)(-2)}{(1)^2}$ | $= \frac{(7)(1)}{(4)^2}$ |
| $= \frac{7}{16}$ | $= \frac{-5}{4}$ | $= -8$ | $= \frac{7}{16}$ |
| f is inc. | f is dec. | f is dec. | f is inc. |




Thus, f is increasing on (– ∞, 0) U (6, ∞) and decreasing on (0, 3) U (3, 6).

**A Critical Point** is all the points (c, f(c)) such that c is in the domain of f and f '(c) = 0 or f '(c) is undefined. The value c is referred to as the critical value.

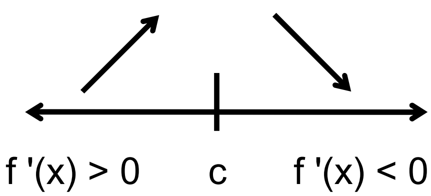
In example #2, x = 0 and x = 6 are critical values, but x = 3 is not since x = 3 is not in the domain of the original function.

The function f(x) has a **relative maximum** at x = c if f(c) ≥ f(x) for all x in the interval (a, b) containing c.

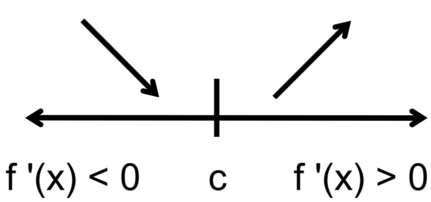
The function f(x) has a **relative minimum** at x = c if f(c) ≤ f(x) for all x in the interval (a, b) containing c.

**First Derivative Test**
Let (c , f(c)) be a critical point (i.e., f '(c) = 0 or is undefined). Then (c, f(c)) is:

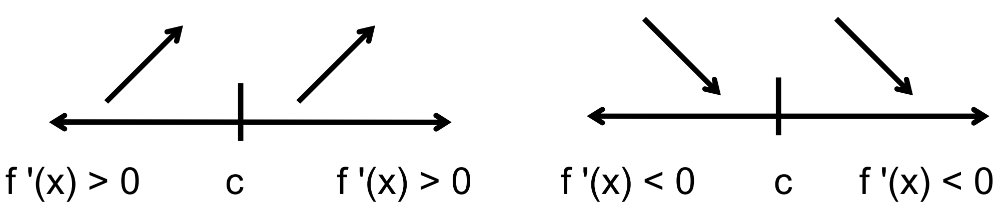
a) a relative maximum if f '(x) > 0 (f is increasing) to the left of c and f '(x) < 0 (f is decreasing) to the right of c.




b) a relative minimum if f '(x) < 0 (f is decreasing) to the left of c and f '(x) > 0 (f is increasing) to the right of c.




c) neither if f '(x) has the same sign on both sides of c.



Find all the relative extrema (i.e., the relative maximum and minimum) of the following:

Ex. 3 $f(x) = 3x^4 - 4x^3$

<u>Solution:</u>

Since f(x) is a polynomial, its domain is all real numbers. Computing the derivative, we get:

$f'(x) = \frac{d}{dx}[3x^4 - 4x^3] = 12x^3 - 12x^2$.

f ' is defined for all real numbers. Setting f '(x) = 0 and solving yields:

$$12x^3 - 12x^2 = 0$$
$$12x^2(x - 1) = 0$$
$$x = 0 \text{ or } x = 1$$

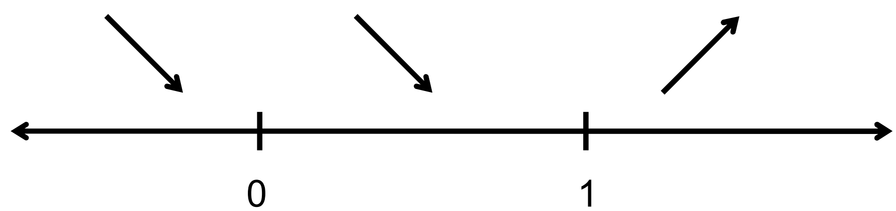
So, x = 0 and x = 1 are the critical values of f. Evaluating the function at these values yields:

$f(0) = 3(0)^4 - 4(0)^3 = 0$

and $f(1) = 3(1)^4 - 4(1)^3 = 3 - 4 = -1$.

Thus, (0, 0) and (1, – 1) are the critical points.

We mark x = 0 and x = 1 on the number line and find where f is increasing and decreasing:

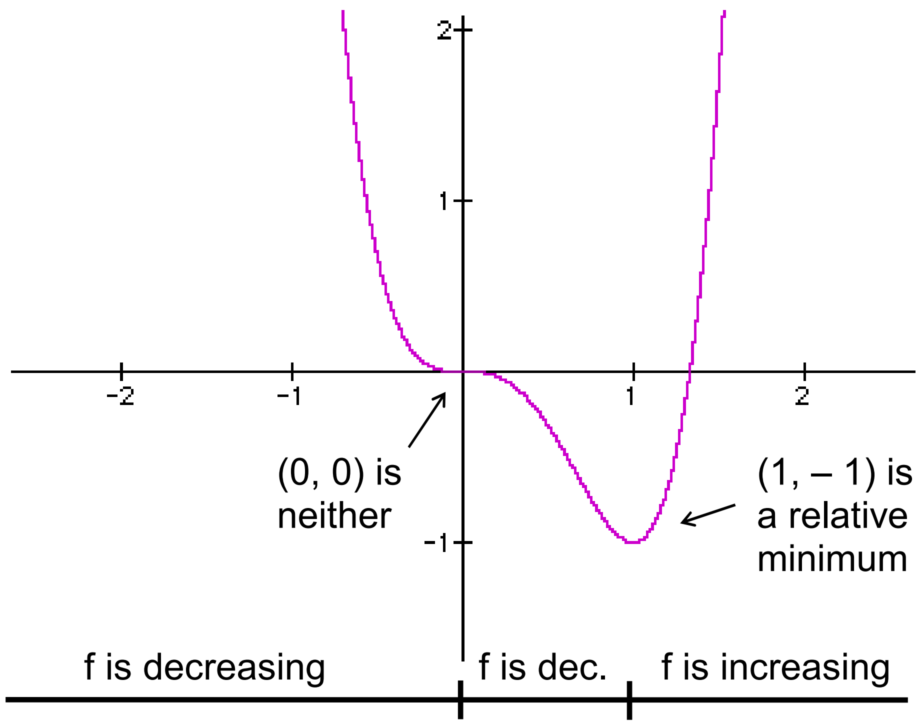



| | | |
|---|---|---|
| Pick | Pick | Pick |
| $x = -1$ | $x = \frac{1}{2}$ | $x = 2$ |
| $f'(-1) =$ | $f'(\frac{1}{2}) =$ | $f'(2) =$ |
| $12(-1)^3 - 12(-1)^2$ | $12(\frac{1}{2})^3 - 12(\frac{1}{2})^2$ | $12(2)^3 - 12(2)^2$ |
| $= -12 - 12$ | $= 1.5 - 3$ | $= 96 - 48$ |
| $= -12$ | $= -1.5$ | $= 48$ |
| f is dec. | f is dec. | f is inc. |

Thus, f is increasing on $(1, \infty)$ & decreasing on $(-\infty, 0) \cup (0, 1)$. By the first derivative test, f has a relative minimum at (1, – 1).

The critical point, (0, 0), is neither a relative maximum or minimum. The function decreases to that point, momentary levels off at (0, 0) and then continues to decrease. Let's look at the graph of f on a graphing calculator in order to get a better sense on the meaning of the critical points.

The graph of f (x) = $3x^4 - 4x^3$.




To get the actual shape of the graph, we will need to determine the concavity of the function f. We will study this in section 3.2.

Ex. 4 g(x) = $\sqrt[3]{(x-3)^2}$

<u>Solution:</u>

Since g(x) is an odd root, the domain of g is all real numbers. We can rewrite the function as g(x) = $(x - 3)^{2/3}$. Computing the derivative, we get:

$g'(x) = \frac{d}{dx}[(x-3)^{2/3}] = \frac{2}{3}(x-3)^{-1/3}(1) = \frac{2}{3\sqrt[3]{x-3}}$.

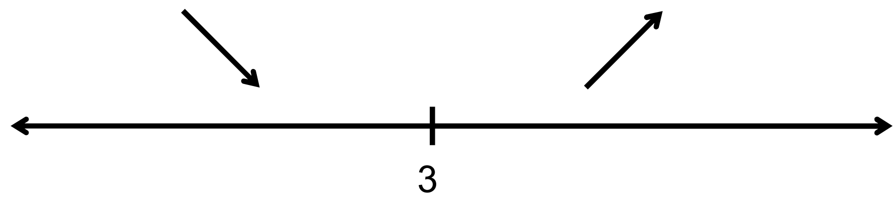
g '(x) is undefined at x = 3 and x = 3 is in the domain of the original function. In fact g(3) = $\sqrt[3]{((3)-3)^2} = \sqrt[3]{0} = 0$. Thus, (3, 0) is a critical point. Setting g '(x) = 0 and solving yields:

$\frac{2}{3\sqrt[3]{x-3}} = 0$ (multiply by $3\sqrt[3]{x-3}$)

$3\sqrt[3]{x-3} \bullet \frac{2}{3\sqrt[3]{x-3}} = 0 \bullet 3\sqrt[3]{x-3}$

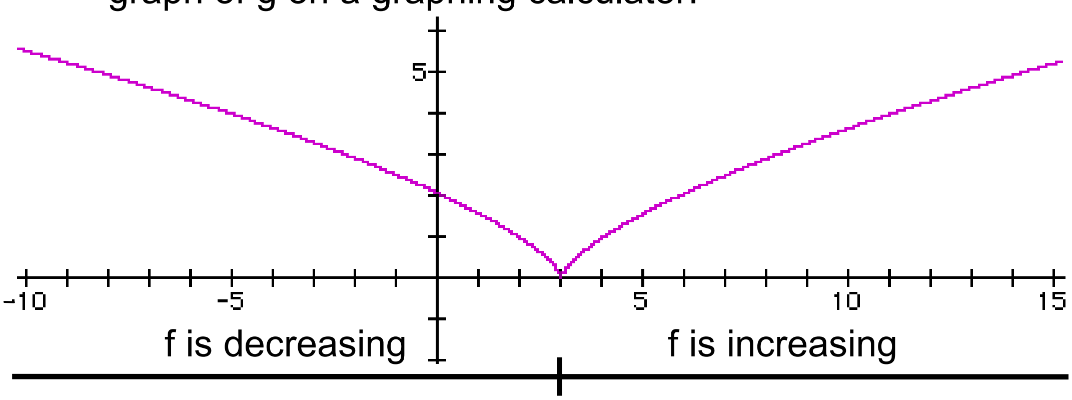
2 = 0, no solution.

Thus, (3, 0) is the only critical point. We mark x = 3 on the number line and find where g is increasing and decreasing:




| Pick | Pick |
|---|---|
| x = 2 | x = 4 |
| g '(2) | g '(4) |
| $= \frac{2}{3\sqrt[3]{2-3}}$ | $= \frac{2}{3\sqrt[3]{4-3}}$ |
| $= \frac{2}{3\sqrt[3]{-1}}$ | $= \frac{2}{3\sqrt[3]{1}}$ |
| $= \frac{2}{3(-1)}$ | $= \frac{2}{3(1)}$ |
| $= -\frac{2}{3}$ | $= \frac{2}{3}$ |

Thus, g is increasing on (3, ∞) and decreasing on (– ∞, 3). By the first derivative test, g has a relative minimum at (3, 0). Since g is defined at x = 3, but g ' is undefined at x = 3, suggest that g either has a sharp corner or a vertical tangent at x = 3. Let's look at the graph of g on a graphing calculator:




Thus, we have a sharp corner at x = 3.