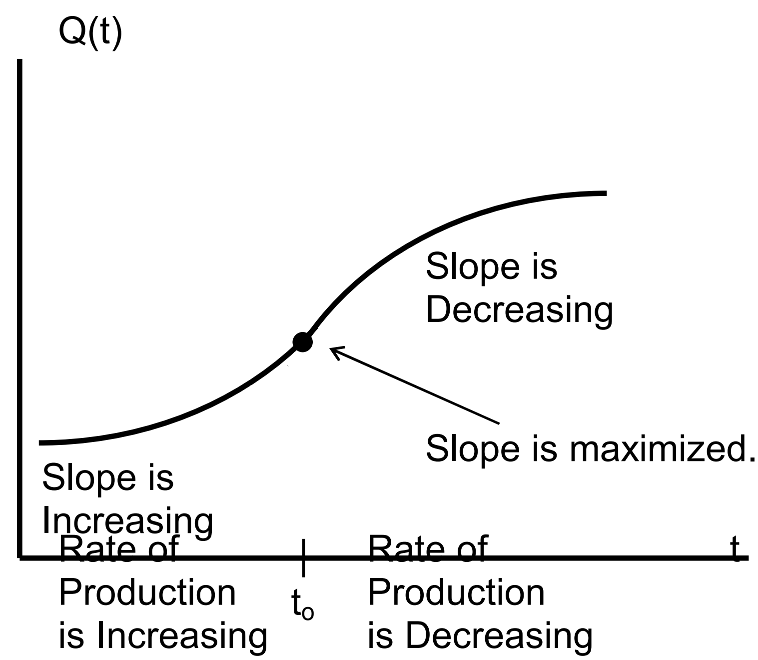
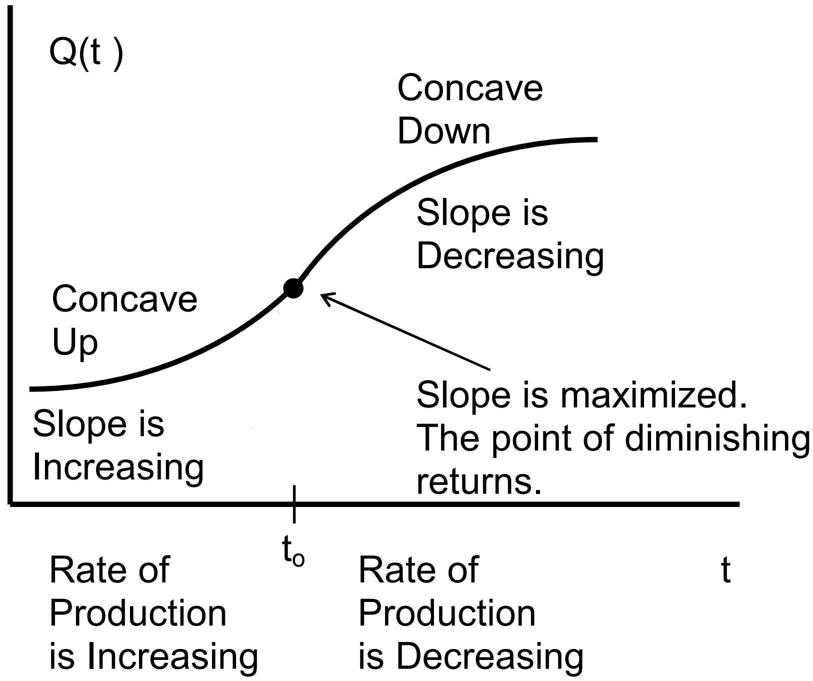
Section 3.2 – The Second Derivative Test and Concavity

The concavity of a function describes the rate of change of the derivative of a function. In other words, it describes where the derivative of the function is increasing and decreasing. Let's consider the following example.

Ex. 1 Let Q(t) be the amount a person produces during an eight hour shift where t is the number of hours into the shift. Looking at the graph below, determine when the work is most efficient (i.e., when is productivity maximized).



Solution:
The productivity is maximized when the slope of the tangent line is maximized. That occurs at $t = t_o$. When the slope of the tangent line is increasing, we say that Q(t) is **Concave up.** When the slope of the tangent line is decreasing, we say that Q(t) is **Concave down.** The point where the productivity is maximized is called the **point of diminishing returns**. Let's look at the graph and see how these ideas relate to the original graph.

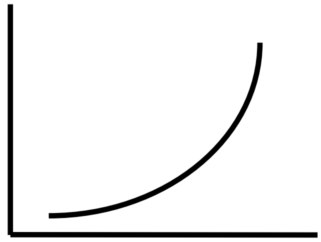


If the rate of change, f '(x), (the slope of the tangent lines) is increasing, then f is concave up. (i.e., $\frac{d}{dx}(f'(x)) > 0$).
If the rate of change, f '(x), (the slope of the tangent lines) is decreasing, then f is concave down. (i.e., $\frac{d}{dx}(f'(x)) < 0$). But
$f''(x) = \frac{d}{dx}(f'(x))$, thus we can make the following definition.

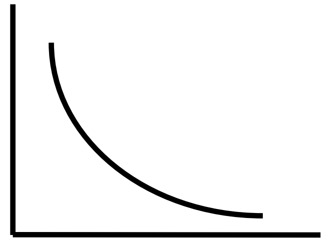
<u>Concavity</u>
If $f''(x) > 0$ for all x in (a, b), then f is concave up on (a, b).
If $f''(x) < 0$ for all x in (a, b), then f is concave down on (a, b).

The concavity of a function is independent of whether a function is increasing or decreasing. Considering the following:

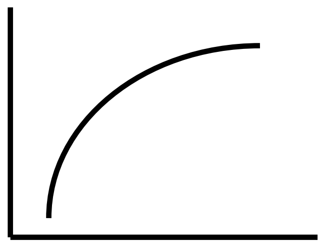
Ex. 2



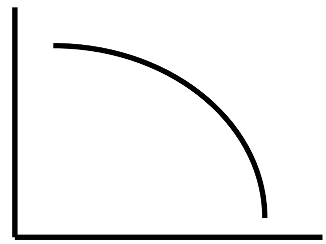
f is increasing and concave up.



f is decreasing and concave up.



f is increasing and concave down.



f is decreasing and concave down.

A point where the concavity changes (look for f "(d) = 0 or f "(d) is undefined) is called an **inflection point**. An inflection point is a "twist" in the graph. Here are some examples of inflection points.

Ex. 3 For the function $f(x) = x^3 - 3x^2 - 24x + 2$,

a) Find where f is increasing and decreasing

b) Find where f is concave up and down.

c) Use the information from parts a and b to sketch the graph of f.

Solution:

a) Since f(x) is a polynomial, its domain is all real numbers. Computing the derivative, we get:

$f'(x) = \frac{d}{dx}[x^3 - 3x^2 - 24x + 2] = 3x^2 - 6x - 24$

f ' is defined for all real numbers. Setting f '(x) = 0 and solving yields:

$3x^2 - 6x - 24 = 0$

$3(x^2 - 2x - 8) = 0$

$3(x - 4)(x + 2) = 0$

$x = 4$ or $x = -2$

So, $x = -2$ and $x = 4$ are the critical values of f.

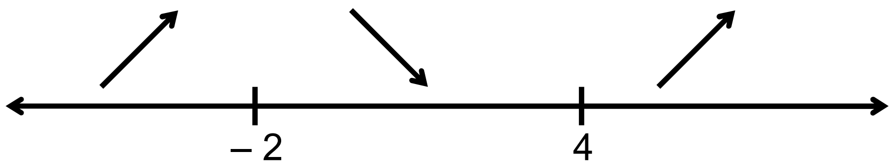
Evaluating the function at these values yields:

$f(-2) = (-2)^3 - 3(-2)^2 - 24(-2) + 2 = 30$

$f(4) = (4)^3 - 3(4)^2 - 24(4) + 2 = -78$

Thus, $(-2, 30)$ and $(4, -78)$ are the critical points.

We mark $x = -2$ and $x = 4$ on the number line and find where f is increasing and decreasing:



Pick	Pick	Pick
$x = -3$	$x = 0$	$x = 5$
$f'(-3) =$	$f'(0) =$	$f'(5) =$
$3(-3)^2 - 6(-3) - 24$	$3(0)^2 - 6(0) - 24$	$3(5)^2 - 6(5) - 24$
$= 27 + 18 - 24$	$= -24$	$= 75 - 30 - 24$
$= 21$		$= 21$
f is inc.	f is dec.	f is inc.

Thus, f is increasing on $(-\infty, -2) \cup (4, \infty)$ & decreasing on $(-2, 4)$.

b) Since $f'(x)$ is a polynomial, its domain is all real numbers. Computing the second derivative, we get:

$f''(x) = \frac{d}{dx}[3x^2 - 6x - 24] = 6x - 6$.

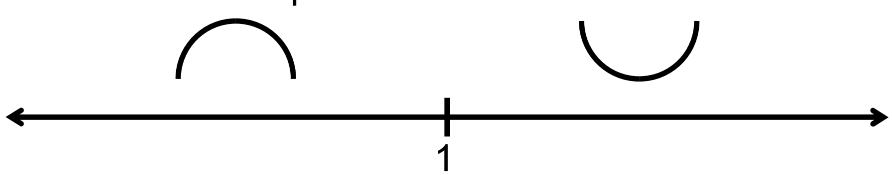
f'' is defined for all real numbers. Setting $f''(x) = 0$ and solving yields:

$$6x - 6 = 0$$
$$x = 1$$

So, f could have a possible inflection point at $x = 1$. Evaluating the function at this value yields:

$f(1) = (1)^3 - 3(1)^2 - 24(1) + 2 = -24$

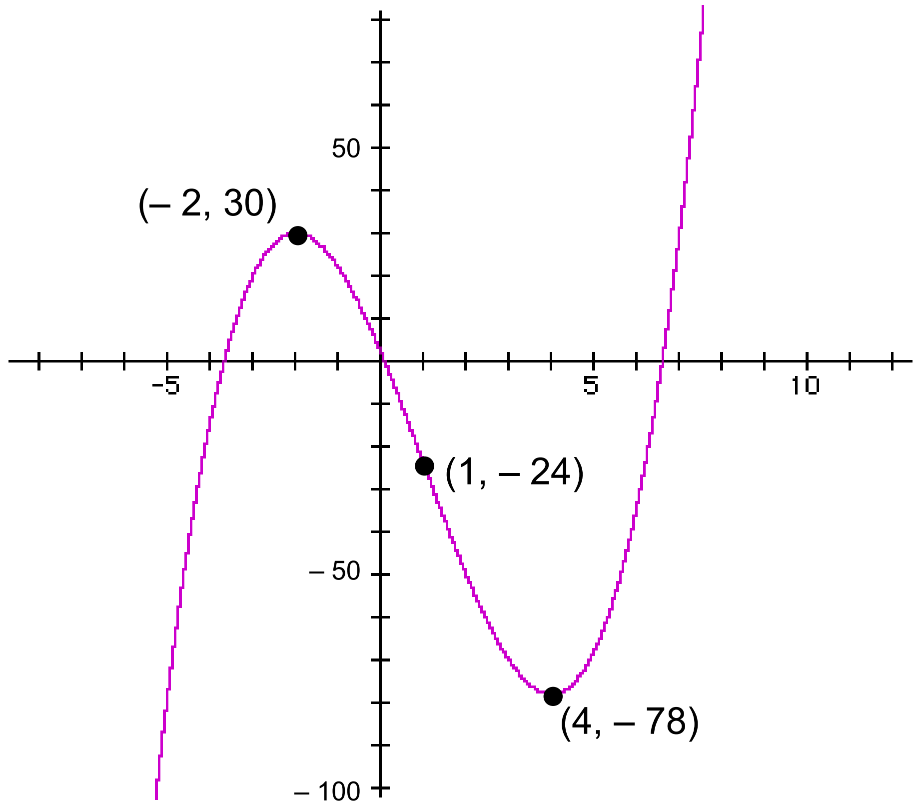
Thus, $(1, -24)$ is the possible inflection point. We mark $x = 1$ on the number line and find where f is concave up and down:

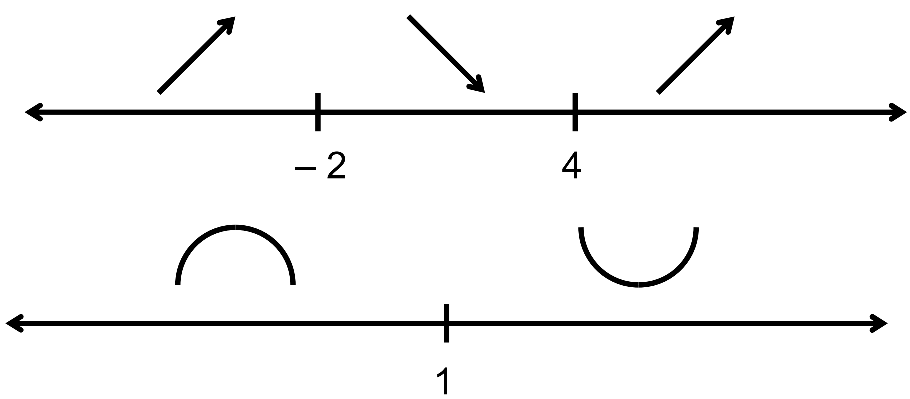


Pick	Pick
$x = 0$	$x = 2$
$f''(0) =$	$f''(2) =$
$6(0) - 6 = -6$	$= 6(2) - 6 = 6$
f is concave down	f is concave up

Thus, f is concave down on $(-\infty, 1)$ and concave up on $(1, \infty)$.

c) To sketch the graph, plot the critical points and the inflection points. Then, under the graph, insert the two sign diagrams obtain from parts a and b.





Now, it is a matter of drawing the graph of the function through the points so that it satisfies the information in the sign diagrams.

In the last example, there was a relative maximum at (– 2, 30) and a relative minimum at (4, – 78). Notice the at (– 2, 30), the function was concave down and at (4, – 78), the function was concave up. The concavity of a function can be used to determine if a critical point is a relative maximum or minimum. This is know as the second derivative test.

<u>The Second Derivative Test:</u>

Suppose (c, f(c)) is a critical point.

1) If $f''(c) > 0$, the f has a relative minimum of f(c) at x = c. (Think positive attitude, smile $\cup$, bottom point)

2) If $f''(c) < 0$, the f has a relative maximum of f(c) at x = c. (Think negative attitude, frown $\cap$, top point)

3) If $f''(c) = 0$ or is undefined, then the test fails. You will have to use the first derivative test.

Ex. 4 Find the relative extrema for $f(x) = (x^2 - 16)^2$.

<u>Solution:</u>

The domain of f is all real numbers. Computing the derivative, we get:

$f'(x) = \frac{d}{dx}[(x^2 - 16)^2] = 2(x^2 - 16)\bullet\frac{d}{dx}[x^2 - 16]$

$= 2(x^2 - 16)(2x) = 4x(x^2 - 16)$.

The derivative is defined for all real numbers. Setting $f'(x) = 0$ and solving yields:

$4x(x^2 - 16) = 0$

$4x(x - 4)(x + 4) = 0$

$x = 0, 4, -4$

Thus, the critical values are x = – 4, 0, and 4. Evaluating f at these values yields:

$f(-4) = ((-4)^2 - 16)^2 = (16 - 16)^2 = 0$.

$f(0) = ((0)^2 - 16)^2 = (-16)^2 = 256$.

$f(4) = ((4)^2 - 16)^2 = (16 - 16)^2 = 0$.

Computing the second derivative, we get:

$f''(x) = \frac{d}{dx}[4x(x^2 - 16)] = \frac{d}{dx}[4x^3 - 64x] = 12x^2 - 64$.

Evaluating f '' at the critical values yields:

$f''(-4) = 12(-4)^2 - 64 = 128 > 0$ $\cup$ - Rel. Min.

$f''(0) = 12(0)^2 - 64 = -64 < 0$ $\cap$ - Rel. Max.

$f''(4) = 12(4)^2 - 64 = 128 > 0$ $\cup$ - Rel. Min.

Thus, f has a relative maximum of 256 at x = 0 and relative minimum of 0 at x = – 4 and at x = 4.

Ex. 5 Find the relative extrema for $f(x) = x + \frac{1}{x}$

Solution:

The domain of f is all real numbers except x = 0.

Computing the derivative, we get:

$f'(x) = \frac{d}{dx}[x + \frac{1}{x}] = \frac{d}{dx}[x + x^{-1}] = 1 - x^{-2} = 1 - \frac{1}{x^2}$.

The derivative is defined for all real numbers except x = 0, but x = 0 is not in the domain of f. Setting $f'(x) = 0$ and solving yields: $1 - \frac{1}{x^2} = 0$ (multiply by x^2)

$$(x^2)\bullet 1 - (x^2)\bullet\frac{1}{x^2} = (x^2)\bullet 0$$

$$x^2 - 1 = 0$$

$$(x - 1)(x + 1) = 0$$

$$x = 1, -1$$

Thus, the critical values are x = – 1 and 1. Evaluating f at these values yields:

$f(-1) = (-1) + \frac{1}{(-1)} = -1 + -1 = -2$.

$f(1) = (1) + \frac{1}{(1)} = 1 + 1 = 2$

Computing the second derivative, we get:

$f''(x) = \frac{d}{dx}[1 - x^{-2}] = 2x^{-3} = \frac{2}{x^3}$.

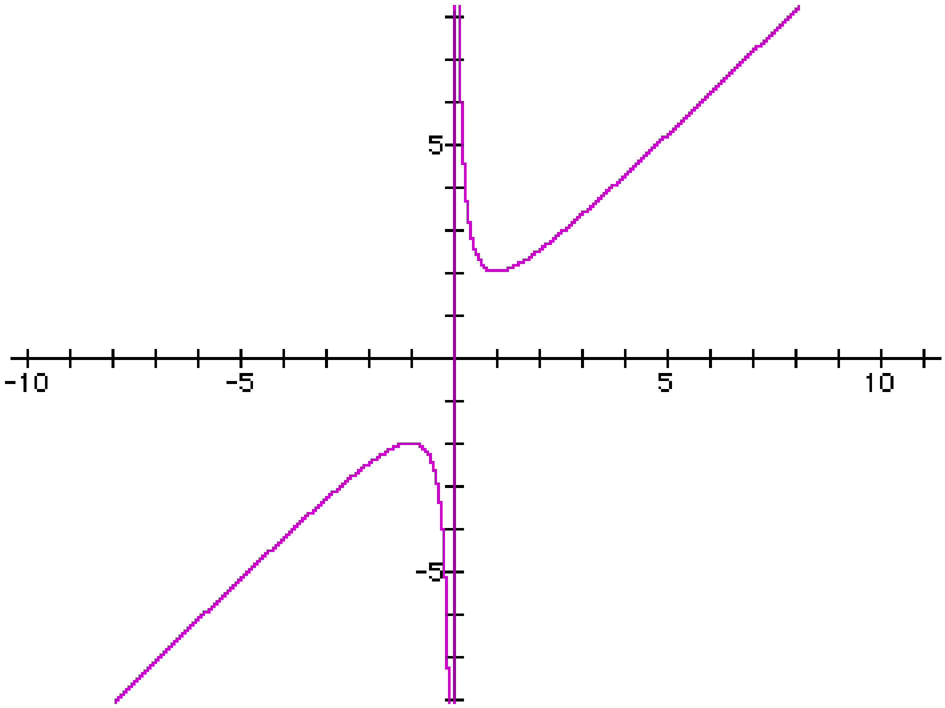
Evaluating f " at the critical values yields:

$f''(-1) = \frac{2}{(-1)^3} = -2 < 0$ $\cap$ - Rel. Max.

$f''(1) = \frac{2}{(1)^3} = 2 > 0$ $\cup$ - Rel. Min.

Thus, f has a relative maximum of – 2 at x = – 1 and relative minimum of 2 at x = 1.

It may seem contradictory that the relative maximum is less than the relative minimum. You may expect the opposite to be true. However, the wild card in this problem is that the function is undefined or discontinuous at x = 0. Let's look at the graph of f:



By looking at the graph of f, the answers that we had in example 5 do make sense. The point (– 1, – 2) is the highest point on the left side of the graph (x < 0) whereas the point
(1, 2) is the lowest point on the right side of the graph (x > 0). The function f has a **Vertical Asymptote** at x = 0 since on the left side, f decreases without bound as x gets close to zero and on the right side, f increases without bound as x gets close to zero. In other words,

$$\lim_{x \to 0^-} f(x) = -\infty \text{ and } \lim_{x \to 0^+} f(x) = \infty$$

The function also happens to have a slant or oblique asymptote of y = x since as x gets very large, $\frac{1}{x}$ gets very small and $y = f(x) \approx x$. We will examine asymptotes in more depth in the next section.