Section 3.3 – Limits Involving Infinity - Asymptotes

We begin our discussion with analyzing limits as x increases or decreases without bound. We will then explore functions that have limits at infinity. Let's consider the following examples:

Ex. 1 Find a) the $\lim_{x\to\infty} \frac{1}{x}$ and b) the $\lim_{x\to-\infty} \frac{1}{x}$

Solution:

a) In this problem, saying $x \to \infty$ means to let x increase without bound. If we make a table of values, we can then see what happens to the function values:

x	1	100	10000	10^6	10^8
$\frac{1}{x}$	1	0.01	0.0001	10^{-6}	10^{-8}

The function values are getting closer and closer to zero.

Thus, $\lim_{x\to\infty} \frac{1}{x} = 0$.

b) In this problem, saying $x \to -\infty$ means to let x decrease without bound. If we make a table of values, we can then see what happens to the function values:

x	-1	-100	-10000	-10^6	-10^8
$\frac{1}{x}$	-1	-0.01	-0.0001	-10^{-6}	-10^{-8}

The function values are getting closer and closer to zero.

Thus, $\lim_{x\to-\infty} \frac{1}{x} = 0$.

In general, if n is a positive number, then

$$\lim_{x\to\infty} \frac{1}{x^n} = 0 \text{ and } \lim_{x\to-\infty} \frac{1}{x^n} = 0.$$

Let's consider some more examples to build on what we have learned here:

Ex. 2 Find the $\lim_{x \to \infty} 6$.

Solution:
Here, our function is a constant function so $\lim_{x \to \infty} 6 = 6$.

Ex. 3 Find the $\lim_{x \to -\infty} x^3 - 6x + 2$.

Solution:
As x decreases without bound, so does $x^3 - 6x + 2$. In other words, as x gets to be a very large negative number, so does $x^3 - 6x + 2$. In fact, the x^3 term will dominate the rest of the expression. So, $\lim_{x \to -\infty} x^3 - 6x + 2$

$= -\infty$.

Ex. 4 Find the $\lim_{x \to -\infty} x^4 - 7x + 2$.

Solution:
As x decreases without bound, $x^4 - 7x + 2$ *increases* without bound since $(-\#)^4 = +$ answer. In other words, as x gets to be a very large negative number, $x^4 - 7x + 2$ becomes a very large positive number since the x^4 term dominates the expression. So, $\lim_{x \to -\infty} x^4 - 7x + 2$

$= \infty$.

Ex. 5 Find the $\lim_{x \to \infty} -3x^2$.

Solution:
As x increases without bound, $-3x^2$ decreases without bound since $-3(+\#)^2 = -$ answer. So, $\lim_{x \to \infty} -3x^2 = -\infty$.

Ex. 6 Find the $\lim_{x \to \infty} \frac{6x^2 - 5x + 2}{7 - 3x^2}$.

Solution:
As x increases without bound, both the numerator and the denominator get large. This does not allow us to determine the limit. But, since x is increasing without

bound, x is staying well away from zero. Thus, we can use the idea that if n is a positive number, then $\lim\limits_{x\to\infty} \frac{1}{x^n} = 0$. First, we find the degree of the polynomial in the denominator. We then multiply the top and bottom of the rational expression by 1/x raised to that power. In this example, we will multiply the top and bottom by $\frac{1}{x^2}$.

After that, we can take the limit:

$$\lim_{x\to\infty} \frac{6x^2-5x+2}{7-3x^2} = \lim_{x\to\infty} \frac{\frac{6x^2}{x^2}-\frac{5x}{x^2}+\frac{2}{x^2}}{\frac{7}{x^2}-\frac{3x^2}{x^2}} = \lim_{x\to\infty} \frac{6-\frac{5}{x}+\frac{2}{x^2}}{\frac{7}{x^2}-3} = \frac{6-0+0}{0-3}$$

$= \frac{6}{-3} = -2$. Thus, $\lim\limits_{x\to\infty} \frac{6x^2-5x+2}{7-3x^2} = -2$.

Ex. 7 Find $\lim\limits_{x\to-\infty} \frac{5x^3-3}{2x^4-3x+2}$.

Solution:

Since the degree of the polynomial in the denominator is 4, will we multiply top and bottom by $\frac{1}{x^4}$ and take the limit:

$$\lim_{x\to-\infty} \frac{5x^3-3}{2x^4-3x+2} = \lim_{x\to-\infty} \frac{\frac{5x^3}{x^4}-\frac{3}{x^4}}{\frac{2x^4}{x^4}-\frac{3x}{x^4}+\frac{2}{x^4}} = \lim_{x\to-\infty} \frac{\frac{5}{x}-\frac{3}{x^4}}{2-\frac{3}{x^3}+\frac{2}{x^4}}$$

$= \frac{0-0}{2-0+0} = \frac{0}{2} = 0.$

Ex. 8 Find the $\lim\limits_{x\to\infty} \frac{3x^2-5x^4}{2x^3+3}$.

Solution:

$$\lim_{x\to\infty} \frac{3x^2-5x^4}{2x^3+3} = \lim_{x\to\infty} \frac{\frac{3x^2}{x^3}-\frac{5x^4}{x^3}}{\frac{2x^3}{x^3}+\frac{3}{x^3}} = \lim_{x\to\infty} \frac{\frac{3}{x}-5x}{2+\frac{3}{x^3}} = -\infty \text{ since}$$

the – 5x term in the numerator decreases without bound as $x \to \infty$. Thus, $\lim\limits_{x\to\infty} \frac{3x^2-5x^4}{2x^3+3} = -\infty$.

We can also have limits at infinity as x approaches a number. Consider the following example:

Ex. 9 Find the $\lim_{x \to -2} \frac{1}{(x+2)^2}$.

Solution:

First consider the left-hand limit. As $x \to -2^-$, $(x + 2)^2$ becomes a smaller and smaller positive number. Thus, $\frac{1}{(x+2)^2}$ becomes a larger and larger positive number or increases without bound. Hence, $\lim_{x \to -2^-} \frac{1}{(x+2)^2} = \infty$.

Now consider the right-hand limit. As $x \to -2^+$, $(x + 2)^2$ becomes a smaller and smaller positive number. Thus, $\frac{1}{(x+2)^2}$ becomes a larger and larger positive number or increases without bound. Hence, $\lim_{x \to -2^+} \frac{1}{(x+2)^2} = \infty$.

Since, $\lim_{x \to -2^-} \frac{1}{(x+2)^2} = \lim_{x \to -2^+} \frac{1}{(x+2)^2}$, then $\lim_{x \to -2} \frac{1}{(x+2)^2} = \infty$.

Ex. 10 Find the $\lim_{x \to -2} \frac{1}{x+2}$.

Solution:

First consider the left-hand limit. As $x \to -2^-$, x + 2 becomes a smaller and smaller negative number. Thus, $\frac{1}{x+2}$ becomes a larger and larger negative number or decreases without bound. Hence, $\lim_{x \to -2^-} \frac{1}{x+2} = -\infty$.

Now consider the right-hand limit. As $x \to -2^+$, x + 2 becomes a smaller and smaller positive number. Thus, $\frac{1}{x+2}$ becomes a larger and larger positive number or increases without bound. Hence, $\lim_{x \to -2^+} \frac{1}{x+2} = \infty$. Since,

$$\lim_{x\to -2^-} \frac{1}{x+2} \neq \lim_{x\to -2^+} \frac{1}{x+2}, \text{ then } \lim_{x\to -2} \frac{1}{x+2} \text{ does not exist.}$$

We now can use the ideas of limits involving infinity to discuss the asymptotes of graphs. The asymptotes of a graph are the "straight lines" that a function approximates as the values of x increase without bound, decrease without bound, or approach a number. We will compare our definition of asymptotes to one that you may have seen in College Algebra. The definitions in Calculus are much broader and work with a wider range of functions. It is very important to be sure that f(x) is in lowest terms.

<u>Vertical Asymptotes:</u>

College Algebra: x = a is a vertical asymptote if

$f(a) = \frac{\text{Non–Zero \#}}{0}$.

Calculus: If $\lim_{x\to a^-} f(x) = \pm\infty$ and/or $\lim_{x\to a^+} f(x) = \pm\infty$, then x = a is a vertical asymptote.

<u>Find the vertical asymptotes of the following:</u>

Ex. 11 $f(x) = \frac{3}{x-2}$.

<u>Solution:</u>

$f(x) = \frac{3}{x-2}$ has a vertical asymptote of x = 2 since

$\lim_{x\to 2^-} \frac{3}{x-2} = -\infty$ (also $\lim_{x\to 2^+} \frac{3}{x-2} = +\infty$).

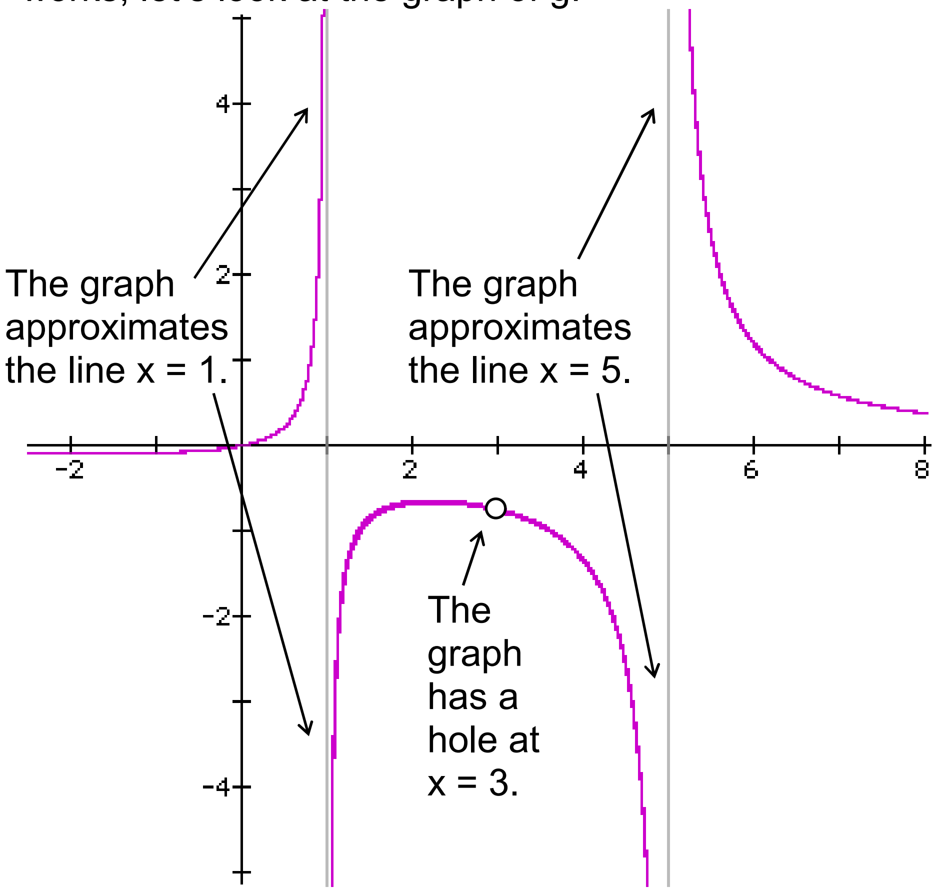
Ex. 12 $g(x) = \frac{x^2-3x}{(x^2-6x+5)(x-3)}$.

<u>Solution:</u>

We first have to check to see if it is reduced to lowest terms: $\frac{x^2-3x}{(x^2-6x+5)(x-3)} = \frac{x(x-3)}{(x-5)(x-1)(x-3)} = \frac{x}{(x-5)(x-1)}$.

Since $\lim_{x\to 5^\pm} \frac{x}{(x-5)(x-1)} = \pm\infty$ and $\lim_{x\to 1^\pm} \frac{x}{(x-5)(x-1)} = \pm\infty$, g has vertical asymptotes of x = 5 and x = 1. Since x – 3 divided out, the function has a hole at x = 3.

In the last example, the function had vertical asymptotes of x = 5 and x = 1. This means that as x get close to either one or five, the function approximates the vertical line. To see how this works, let's look at the graph of g:



Horizontal Asymptotes:

College Algebra:

1) If the degree of the polynomial in the denominator is equal to the degree of the polynomial in the numerator, then $y = \frac{a}{b}$ is a horizontal asymptote where a and b are the leading coefficients of the polynomials in the numerator and denominator respectively.

2) If the degree of the polynomial in the denominator is greater than the degree of the polynomial in the numerator, then y = 0 is a horizontal asymptote.

3) If the degree in the polynomial in the numerator is larger than the degree of the polynomial in the denominator, then there is no horizontal asymptote.

Calculus:

If $\lim_{x \to \infty} f(x) = R$, and/or $\lim_{x \to -\infty} f(x) = R$, where $R \neq \pm \infty$, then y = R is a horizontal asymptote.

Find the Horizontal Asymptotes of the following:

Ex. 13 $h(x) = \frac{6x^2 - 5x + 3}{2 + 6x - 7x^2}$.

Solution:

Since $\lim_{x \to \infty} \frac{6x^2 - 5x + 3}{2 + 6x - 7x^2} = \lim_{x \to \infty} \left(\frac{6x^2 - 5x + 3}{2 + 6x - 7x^2}\right) \bullet \frac{\frac{1}{x^2}}{\frac{1}{x^2}}$

$= \lim_{x \to \infty} \frac{\frac{6x^2}{x^2} - \frac{5x}{x^2} + \frac{3}{x^2}}{\frac{2}{x^2} + \frac{6x}{x^2} - \frac{7x^2}{x^2}} = \lim_{x \to \infty} \frac{6 - \frac{5}{x} + \frac{3}{x^2}}{\frac{2}{x^2} + \frac{6}{x} - 7} = \frac{6 - 0 + 0}{0 + 0 - 7} = -\frac{6}{7}$, then

h has a horizontal asymptote of $y = -\frac{6}{7}$. Also,

$\lim_{x \to -\infty} \frac{6x^2 - 5x + 3}{2 + 6x - 7x^2} = -\frac{6}{7}$.

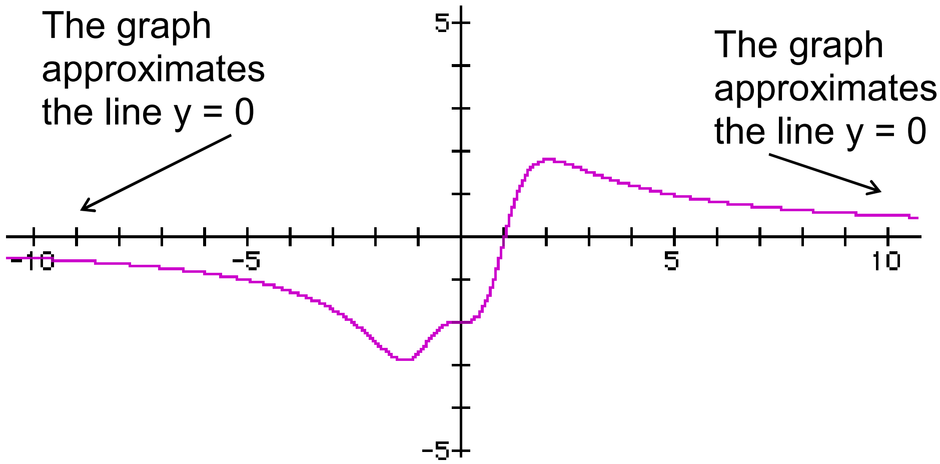
Ex. 14 $f(x) = \frac{5x^3 - 6}{x^4 + 3}$.

Solution:

Since $\lim_{x \to \infty} \frac{5x^3 - 6}{x^4 + 3} = \lim_{x \to \infty} \left(\frac{5x^3 - 6}{x^4 + 3}\right) \bullet \frac{\frac{1}{x^4}}{\frac{1}{x^4}} = \lim_{x \to \infty} \frac{\frac{5x^3}{x^4} - \frac{6}{x^4}}{\frac{x^4}{x^4} + \frac{3}{x^4}}$

$= \lim_{x \to \infty} \frac{\frac{5}{x} - \frac{6}{x^4}}{1 + \frac{3}{x^4}} = \frac{0 - 0}{1 + 0} = \frac{0}{1} = 0$ (and $\lim_{x \to -\infty} \frac{5x^3 - 6}{x^4 + 3} = 0$), then f has a horizontal asymptote of y = 0.

In the last example, the function had a horizontal asymptote of y = 0. This means that as x get increases or decreases without bound, the function approximates the horizontal line y = 0. To see how this works, let's look at the graph of f:



Slant (Oblique) asymptotes:

College Algebra:

If the degree of the polynomial in the numerator is one more than the degree of the polynomial in the denominator, then divide. The result is f(x) = mx + b + h(x).

Thus, y = mx + b is a slant asymptote.

Calculus:

If f(x) = mx + b + h(x) where $\lim\limits_{x\to\infty} h(x) = 0$, and/or $\lim\limits_{x\to-\infty} h(x) = 0$, then y = mx + b is the slant asymptote.

Note: A function cannot have both a horizontal and a slant asymptote.

Find the slant asymptote of the following:

Ex. 15 $y = \frac{x^2 - 5x + 2}{x - 3}$

Solution:

We will use synthetic division to divide:

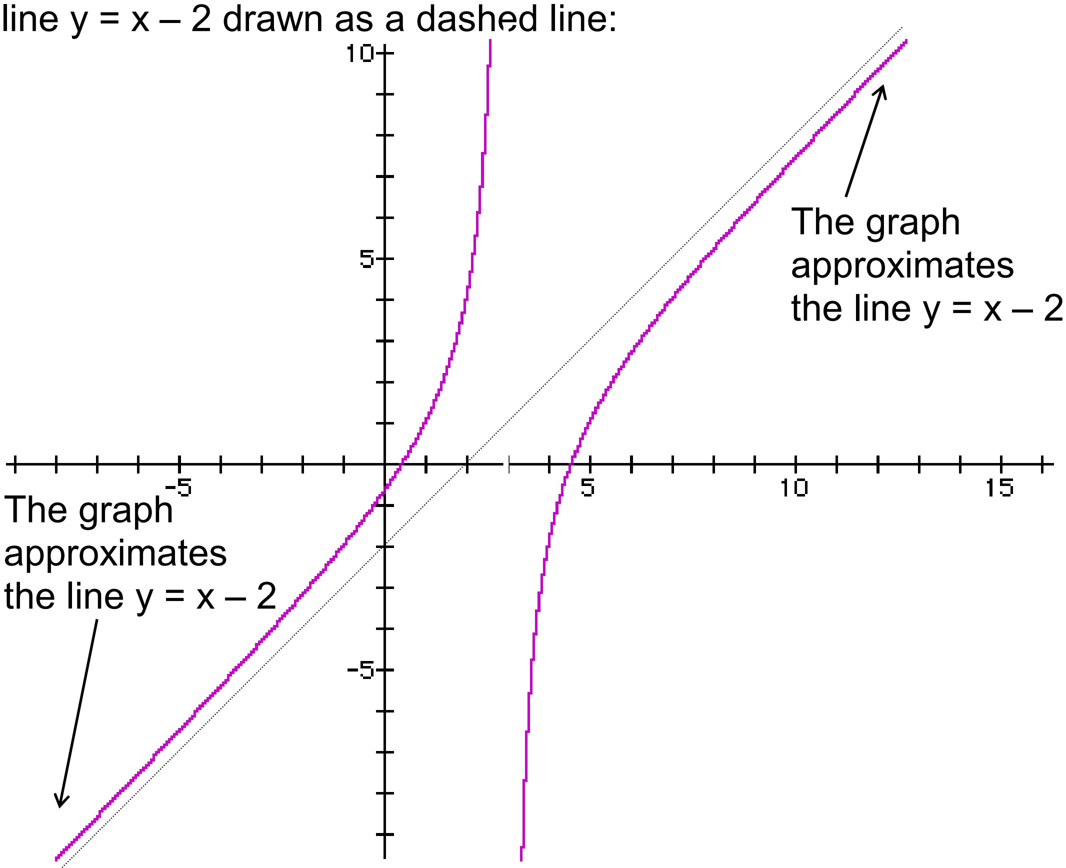
3	1	– 5	2
		3	– 6
	1	– 2	– 4

Thus, $y = \frac{x^2 - 5x + 2}{x - 3} = x - 2 + \frac{-4}{x - 3}$. Since $\lim\limits_{x\to\infty} \frac{-4}{x - 3}$

$$= \lim_{x \to \infty} \left(\frac{-4}{x-3}\right) \bullet \frac{\frac{1}{x}}{\frac{1}{x}} = \lim_{x \to \infty} \frac{\frac{-4}{x}}{\frac{x}{x} - \frac{3}{x}} = \lim_{x \to \infty} \frac{\frac{-4}{x}}{1 - \frac{3}{x}} = \frac{0}{1-0} = \frac{0}{1} = 0,$$

then the function has a slant asymptote of y = x – 2.

In the last example, the function had a slant asymptote of y = x – 2. This means that as x get increases or decreases without bound, the function approximates the line y = x – 2. To see how this works, let's look at the graph of function with the line y = x – 2 drawn as a dashed line:



Now, we will put all the techniques that we have learned thus far in this chapter together and outline a general procedure that we will follow when graphing functions. The procedure is outlined on the next page and afterwards, we will use it to graph some examples.

Steps for Graphing f(x)

1) Find the domain of f. Reduce f(x) to lowest terms and identify any "holes" in the function.

2) Find the y-intercept and, if they are relatively easy to find, the x-intercept(s).

3) Find f '(x) and f "(x).

4) Find all asymptotes of f(x):

   I) Vertical Asymptotes:
   x = a is a vertical asymptote if $\lim_{x \to a^+} f(x) = \pm\infty$ and/or $\lim_{x \to a^-} f(x) = \pm\infty$.

   II) Horizontal Asymptotes:
   y = b is a horizontal asymptote if $\lim_{x \to \infty} f(x) = b$ and/or $\lim_{x \to -\infty} f(x) = b$.

   III) Slant Asymptotes:
   y = mx + b is a slant asymptote if f(x) = mx + b + g(x) where $\lim_{x \to \infty} g(x) = 0$ and/or $\lim_{x \to -\infty} g(x) = 0$.

5) Find all the critical points, i.e., all the values c of the domain of f where f '(c) = 0 and undefined. Determine where the function is increasing (f '(x) > 0) or decreasing (f '(x) < 0).

6) Use the second the derivative test to verify which of the critical points are relative maximums and minimums (If f "(c) > 0, minimum. If f "(c) < 0, maximum.) If the test fails (f "(c) = 0 or is undefined) use the first derivative test:

   I) If f '(x) < 0 to the left of c and f '(x) > 0 to the right of c, then f(c) is a relative minimum.

   II) If f '(x) > 0 to the left of c and f '(x) < 0 to the right of c, then f(c) is a relative maximum.

7) Find all the inflection points, i.e., all the values d of the domain of f where f "(d) = 0 and undefined. Determine the concavity (If f "(x) < 0, down. If f "(x) > 0, up).

8) Sketch the graph. Feel free to plot some additional point

Graph the following:

Ex. 16 $f(x) = 3x^4 - 16x^3 + 18x^2$

Solution:

I) Since f is a polynomial, the domain is $(-\infty, \infty)$.

II) Since $f(0) = 3(0)^4 - 16(0)^3 + 18(0)^2 = 0$, the y-intercept is (0, 0). Setting f(x) = 0 and solving yields:

$3x^4 - 16x^3 + 18x^2 = 0$

$x^2(3x^2 - 16x + 18) = 0$

$x = 0$ or $x = \frac{16 \pm \sqrt{(-16)^2 - 4(3)(18)}}{2(3)}$

$x = 0$ or $x = \frac{8 \pm \sqrt{10}}{3} \approx 1.61$ or 3.72.

Thus, the x-intercepts are (0, 0), (1.61, 0), and (3.72, 0).

III) $f'(x) = \frac{d}{dx}[3x^4 - 16x^3 + 18x^2] = 12x^3 - 48x^2 + 36x$.

$f''(x) = \frac{d}{dx}[12x^3 - 48x^2 + 36x] = 36x^2 - 96x + 36$.

IV) f(x) is a polynomial so it has no asymptotes.

V) $f'(x) = 12x^3 - 48x^2 + 36x$ is a polynomial so it is defined for all real numbers. Setting $f'(x) = 0$ and solving yields:

$12x^3 - 48x^2 + 36x = 0$

$12x(x^2 - 4x + 3) = 0$

$12x(x - 3)(x - 1) = 0$

$x = 0$, $x = 3$, and $x = 1$

Thus, x = 0, 1, and 3 are the critical values.

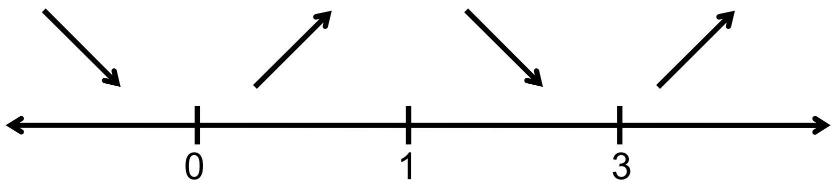
Evaluating f at these values yields:

$f(0) = 3(0)^4 - 16(0)^3 + 18(0)^2 = 0$

$f(1) = 3(1)^4 - 16(1)^3 + 18(1)^2 = 5$

$f(3) = 3(3)^4 - 16(3)^3 + 18(3)^2 = -27$

So, (0, 0), (1, 5), and (3, – 27) are the critical points. Marking x = 0, 1, and 3 on the number line, we can find where f is increasing and decreasing:



Pick – 1	Pick 0.5	Pick 2	Pick 4
f '(– 1)	f '(0.5)	f '(2)	f '(4)
= – 96	= 7.5	= – 24	= 144

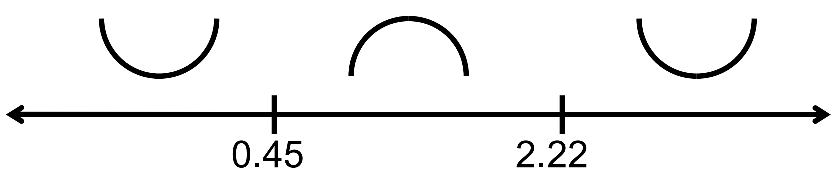
Thus, f is increasing on (0, 1) U (3, ∞) and decreasing on (– ∞, 0) U (1, 3).

VI) f "(0) = $36(0)^2 - 96(0) + 36 = 36$ so (0, 0) is a relative minimum (verifies the results in part V).
f "(1) = $36(1)^2 - 96(1) + 36 = -24$ so (1, 5) is a relative maximum (verifies the results in part V).
f "(3) = $36(3)^2 - 96(3) + 36 = 72$ so (3, – 27) is a relative minimum (verifies the results in part V).

VII) f "(x) = $36x^2 - 96x + 36$ is a polynomial so it is defined for all real numbers. Setting f "(x) = 0 and solving yields:

$$36x^2 - 96x + 36 = 0$$
$$12(3x^2 - 8x + 3) = 0$$
$$x = \frac{8 \pm \sqrt{(-8)^2 - 4(3)(3)}}{2(3)} = \frac{4 \pm \sqrt{7}}{3} \approx 0.45 \text{ or } 2.22.$$

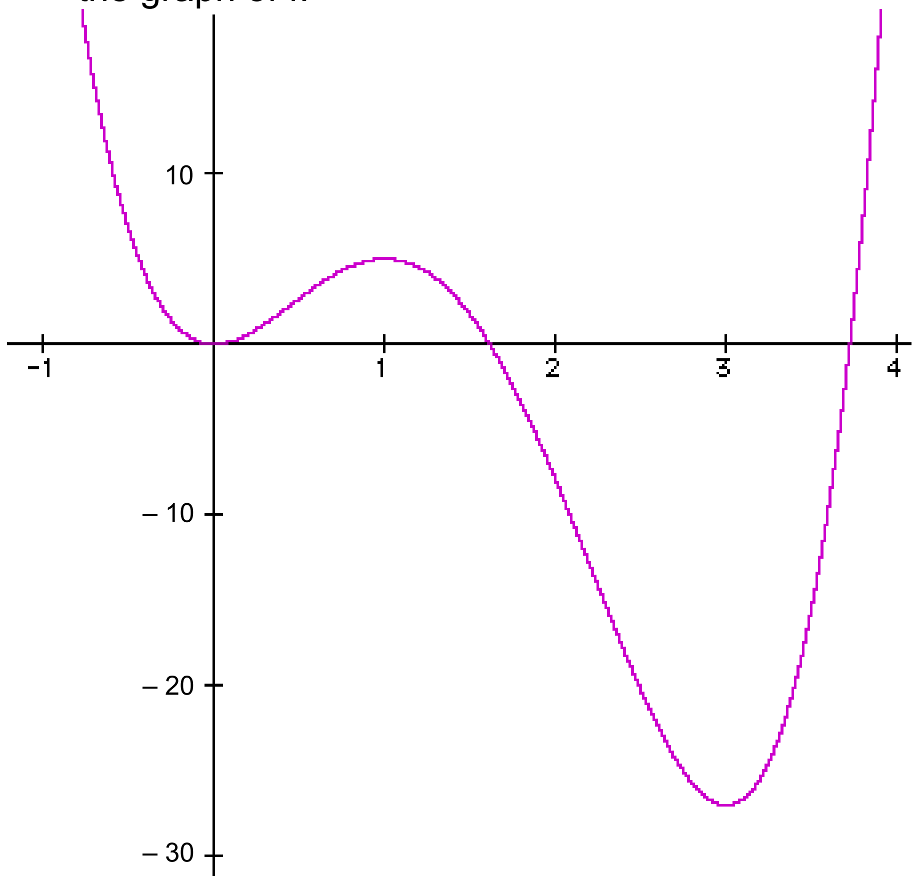
Evaluating f(x) at these values yields: f(0.45) ≈ 2.32 and f(2.22) ≈ – 13.36. Thus, (0.45, 2.32) and (2.22, – 13.36) are the possible inflection points. Marking x ≈ 0.45 and x ≈ 2.22 on the number line, we can find where f is concave up or down:



Pick 0	Pick 1	Pick 3
f "(0) = 36	f "(1) = – 24	f "(3) = 72

Thus, f is concave up on (– ∞, 0.45) U (2.22, ∞) and concave down on (0.45, 2.22).

VIII) Now use the information from parts I - VII to sketch the graph of f:



Ex. 17 $f(x) = \frac{x}{x^2+1}$

Solution:

I) Since $x^2 + 1 \neq 0$, then the domain of f is $(-\infty, \infty)$.

II) Since $f(0) = \frac{(0)}{(0)^2+1} = \frac{0}{1} = 0$, the y-intercept is (0, 0).

Setting f(x) = 0 and solving yields:

$\frac{x}{x^2+1} = 0$ (multiply by $x^2 + 1$)

$(x^2 + 1)\bullet \frac{x}{x^2+1} = 0\bullet(x^2 + 1)$

$x = 0$.

So, the only x-intercept is (0, 0).

III) $f'(x) = \frac{d}{dx}[\frac{x}{x^2+1}] = \frac{(x^2+1)\cdot\frac{d}{dx}[x] - x\cdot\frac{d}{dx}[x^2+1]}{(x^2+1)^2}$

$= \frac{(x^2+1)[1] - x\cdot[2x]}{(x^2+1)^2} = \frac{x^2+1-2x^2}{(x^2+1)^2} = \frac{-x^2+1}{(x^2+1)^2}$.

$f''(x) = \frac{d}{dx}[\frac{-x^2+1}{(x^2+1)^2}]$

$= \frac{(x^2+1)^2\cdot\frac{d}{dx}[-x^2+1] - (-x^2+1)\cdot\frac{d}{dx}[(x^2+1)^2]}{(x^2+1)^4}$

$= \frac{(x^2+1)^2\cdot[-2x] - (-x^2+1)\cdot[2(x^2+1)]\bullet\frac{d}{dx}(x^2+1)}{(x^2+1)^4}$

$= \frac{(x^2+1)^2\cdot[-2x] + 2(x^2-1)\cdot(x^2+1)\bullet(2x)}{(x^2+1)^4}$

$= \frac{(x^2+1)[(x^2+1)\cdot[-2x] + 2(x^2-1)(2x)]}{(x^2+1)^4}$

$= \frac{(x^2+1)[-2x^3-2x+4x^3-4x]}{(x^2+1)^4}$

$= \frac{[2x^3-6x]}{(x^2+1)^3} = \frac{2x(x^2-3)}{(x^2+1)^3}$.

IV) The function has no vertical asymptotes. But,

$$\lim_{x\to\pm\infty}\frac{x}{x^2+1} = \lim_{x\to\pm\infty}\frac{\frac{x}{x^2}}{\frac{x^2}{x^2}+\frac{1}{x^2}} = \lim_{x\to\pm\infty}\frac{\frac{1}{x}}{1+\frac{1}{x^2}} = \frac{0}{1+0} = 0.$$

Thus, f has a horizontal asymptote of y = 0.

V) Since $x^2 + 1 \neq 0$, $f'(x)$ is defined for all real numbers. Setting $f'(x) = 0$ and solving yields:

$\frac{-x^2+1}{(x^2+1)^2} = 0$ (multiply by $(x^2+1)^2$)

$\mathbf{(x^2+1)^2}\cdot\frac{-x^2+1}{(x^2+1)^2} = 0\cdot\mathbf{(x^2+1)^2}$

$-x^2 + 1 = 0$

$-(x^2 - 1) = 0$

$-(x - 1)(x + 1) = 0$

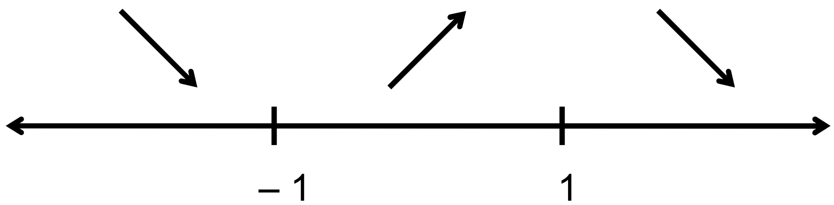
$x = 1$ and -1.

Evaluating f at these values yields:

$f(-1) = \frac{(-1)}{(-1)^2+1} = -\frac{1}{2}$ and $f(1) = \frac{(1)}{(1)^2+1} = \frac{1}{2}$.

So, $(-1, -\frac{1}{2})$ and $(1, \frac{1}{2})$ are the critical points.

Marking x = – 1 and x = 1 on the number line, we can find where f is increasing and decreasing:



Pick – 2 | Pick 0 | Pick 2
f '(– 2) = – 0.12 | f '(0) = 1 | f '(2) = – 0.12

Thus, f is increasing (– 1, 1) and decreasing on $(-\infty, -1) \cup (1, \infty)$.

VI) $f''(-1) = \frac{2(-1)((-1)^2-3)}{((-1)^2+1)^3} = \frac{4}{8} = \frac{1}{2}$ so $(-1, -\frac{1}{2})$ is a relative minimum (verifies the results in part V).

$f''(1) = \frac{2(1)((1)^2-3)}{((1)^2+1)^3} = -\frac{4}{8} = -\frac{1}{2}$ so $(1, \frac{1}{2})$ is a relative maximum (verifies the results in part V).

VII) Since $x^2 + 1 \neq 0$, then f "(x) is defined for all real numbers. Setting f "(x) = 0 and solving yields:

$\frac{2x(x^2-3)}{(x^2+1)^3} = 0$ (multiply by $(x^2 + 1)^3$)

$$\mathbf{(x^2+1)^3} \bullet \frac{2x(x^2-3)}{(x^2+1)^3} = \mathbf{(x^2+1)^3} \bullet 0$$

$2x(x^2 - 3) = 0$

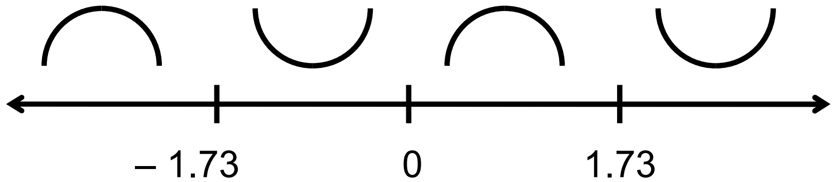
$2x = 0$ or $x^2 - 3 = 0$

$x = 0$ or $x = \pm\sqrt{3} \approx \pm 1.73$.

Evaluating f at these values yields:

f(– 1.73) ≈ – 0.43, f(0) = 0, and f(1.73) ≈ 0.43.

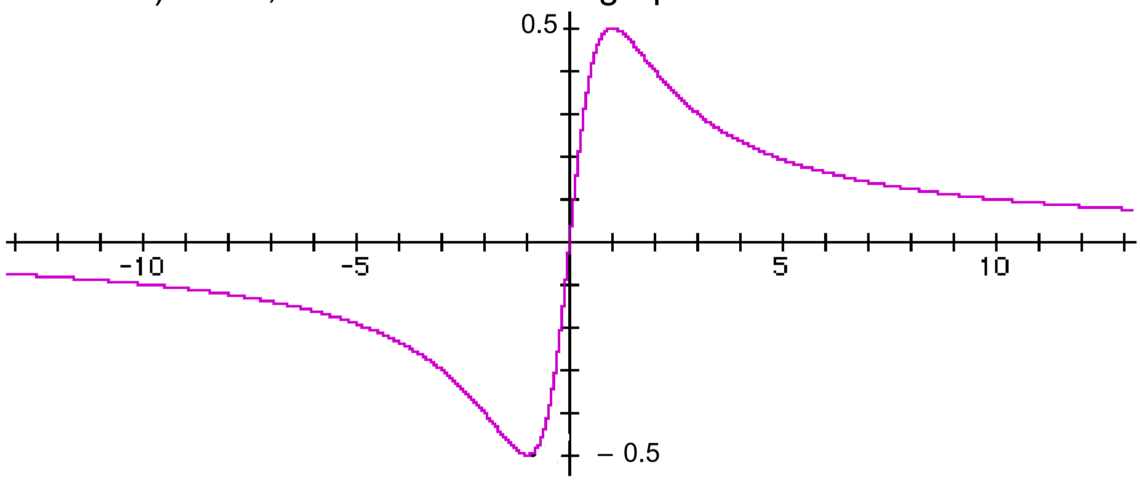
Thus, (– 1.73, – 0.43), (0, 0), (1.73, 0.43) are the possible inflection points. Marking these values on the number line, we can find the concavity of f:



Pick – 2	Pick – 1	Pick 1	Pick 2
f "(– 2)	f "(– 1)	f "(1)	f "(2)
= – 0.32	= 0.5	= – 0.5	= 0.32

Hence, f is concave up on (– 1.73, 0) U (1.73, ∞) and concave down on (– ∞, – 1.73) U (0, 1.73).

VIII) Now, we can sketch the graph:



Ex. 18 $f(x) = \frac{2x+1}{x}$

<u>Solution:</u>

I) Since f is a rational function, it is undefined when the denominator is 0 or when x = 0. Thus, the domain is (– ∞, 0) U (0, ∞).

II) Since f(0) is undefined, there is no y-intercept. Setting f(x) = 0 and solving yields:

$\frac{2x+1}{x} = 0$ (multiply by x)

$\mathbf{x} \cdot \frac{2x+1}{x} = \mathbf{x} \cdot 0$

$2x + 1 = 0$

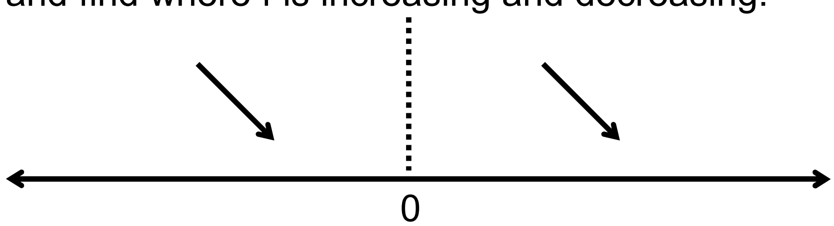
$x = -\frac{1}{2}$.

Thus, $(-\frac{1}{2}, 0)$ is the x - intercept.

III) First rewrite f(x) as $\frac{2x+1}{x} = 2 + \frac{1}{x} = 2 + x^{-1}$. Thus,
$f'(x) = \frac{d}{dx}[2 + x^{-1}] = -x^{-2} = -\frac{1}{x^2}$ and
$f''(x) = \frac{d}{dx}[-x^{-2}] = 2x^{-3} = \frac{2}{x^3}$.

IV) Since $\lim_{x\to 0^{\pm}} 2 + \frac{1}{x} = \pm\infty$, f has a vertical asymptote of x = 0. Also, since $\lim_{x\to\pm\infty} 2 + \frac{1}{x} = 2 + 0 = 2$, f has a horizontal asymptote of y = 2.

V) f '(x) is undefined at x = 0, but x = 0 is not in the domain of f. Thus, x = 0 is not a critical value.
Setting f '(x) = 0 and solving yields:
$-\frac{1}{x^2} = 0$ (multiply by x^2)
$\mathbf{x^2}\cdot(-\frac{1}{x^2}) = \mathbf{x^2}\cdot 0$
– 1 = 0, no solution.
Thus, f has no critical points. Hence, we mark only the value where f is undefined on the number line and find where f is increasing and decreasing:



Pick x = – 1 Pick x = 1
f '(– 1) = – 1 f '(1) = – 1

Thus, f is decreasing on $(-\infty, 0) \cup (0, \infty)$ and increasing nowhere.

VI) Since there are no critical points, this part is not applicable.

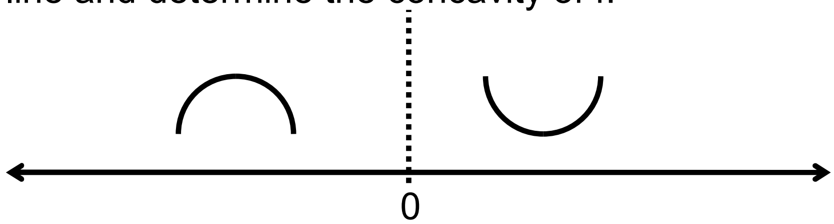
VII) f "(x) is undefined at x = 0, but x = 0 is not in the domain of f. Thus, x = 0 is not to be considered.
Setting f "(x) = 0 and solving yields:

$\frac{2}{x^3} = 0$ (multiply by x^3)

$\mathbf{x^3} \bullet (\frac{2}{x^3}) = \mathbf{x^3} \bullet 0$

2 = 0, no solution.

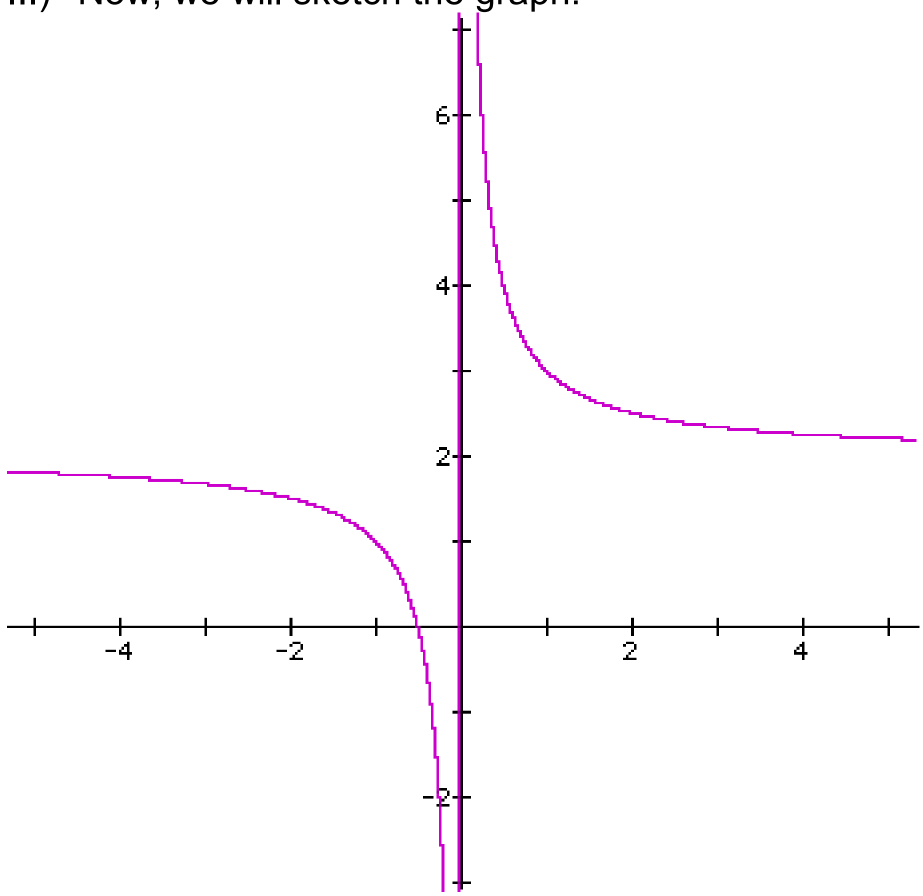
Thus, f has no inflection points. Hence, we mark only the value where f is undefined on the number line and determine the concavity of f:



Pick – 1 | Pick 1
f "(– 1) = – 2 | f "(1) = 2

Thus, f is concave up on (0, ∞) and concave down on (– ∞, 0).

VIII) Now, we will sketch the graph:



Ex. 19 $f(x) = x + \frac{9}{x}$

<u>Solution:</u>

I) Since f is a rational function, it is undefined when the denominator is 0 or when x = 0. Thus, the domain is $(-\infty, 0) \cup (0, \infty)$.

II) Since f(0) is undefined, there is no y-intercept. Setting f(x) = 0 and solving yields:

$x + \frac{9}{x} = 0$ (multiply by x)

$\mathbf{x} \bullet x + \mathbf{x} \bullet \frac{9}{x} = \mathbf{x} \bullet 0$

$x^2 + 9 = 0$

$x^2 = -9$

$x = \pm\sqrt{-9}$, no real solution.

Thus, there are no x - intercepts.

III) First rewrite f(x) as $x + \frac{9}{x} = x + 9x^{-1}$. Thus,

$f'(x) = \frac{d}{dx}[x + 9x^{-1}] = 1 - 9x^{-2} = 1 - \frac{9}{x^2}$ and

$f''(x) = \frac{d}{dx}[1 - 9x^{-2}] = 18x^{-3} = \frac{18}{x^3}$.

IV) Since $\lim\limits_{x \to 0^{\pm}} x + \frac{9}{x} = \pm\infty$, f has a vertical asymptote of x = 0. Also, since $\lim\limits_{x \to \pm\infty} \frac{9}{x} = 0$, f has a slant asymptote of y = x.

V) f '(x) is undefined at x = 0, but x = 0 is not in the domain of f. Thus, x = 0 is not a critical value. Setting f '(x) = 0 and solving yields:

$1 - \frac{9}{x^2} = 0$ (multiply by x^2)

$\mathbf{x^2} \bullet 1 - \mathbf{x}^2 \bullet \frac{9}{x^2} = \mathbf{x^2} \bullet 0$

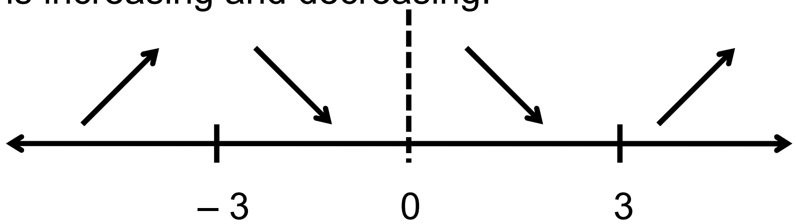
$x^2 - 9 = 0$

$(x - 3)(x + 3) = 0$

$x = 3$ or $x = -3$.

Thus, x = – 3 and x = 3 are the critical values.

Evaluating f at these values yields:
$f(3) = (3) + \frac{9}{(3)} = 6$ and $f(-3) = (-3) + \frac{9}{(-3)} = -6$.
Thus, (3, 6) and (– 3, – 6) are the critical points. Marking these values and the value where f is undefined on the number line, we can find where f is increasing and decreasing:

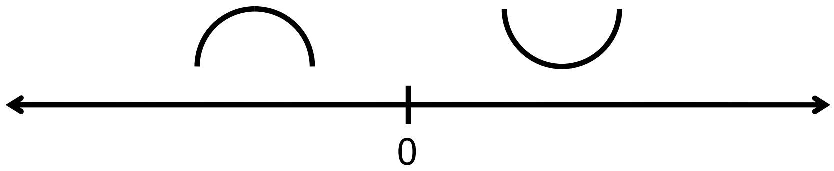


Pick – 4	Pick – 1	Pick 1	Pick 4
$f'(-4)$	$f'(-1)$	$f'(1)$	$f'(4)$
$= \frac{7}{16}$	$= -8$	$= -8$	$= \frac{7}{16}$

Thus, f is increasing on $(-\infty, -3) \cup (3, \infty)$ and decreasing on $(-3, 0) \cup (0, 3)$.

VI) $f''(-3) = \frac{18}{(-3)^3} = -\frac{2}{3}$ so (– 3, – 6) is a relative maximum (verifies the results in part V).
$f''(3) = \frac{18}{(3)^3} = \frac{2}{3}$ so (3, 6) is a relative minimum (verifies the results in part V).

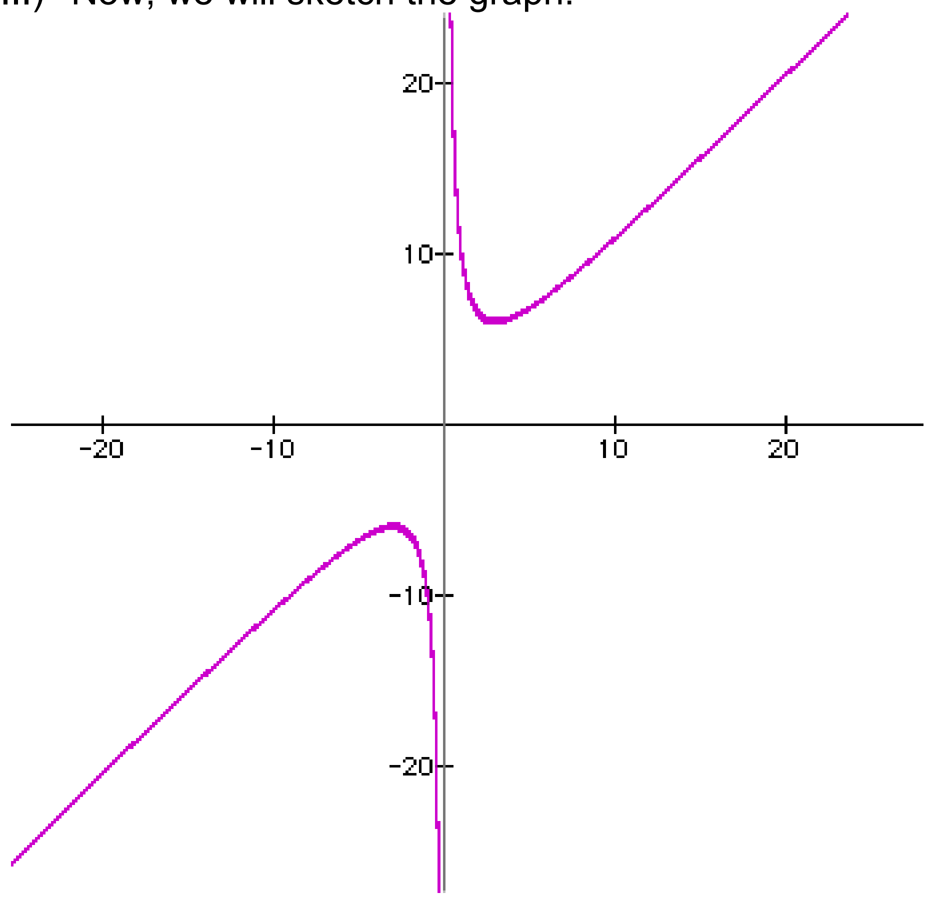
VII) $f''(x)$ is undefined at x = 0, but x = 0 is not in the domain of f. Thus, x = 0 is not to be considered.
Setting $f''(x) = 0$ and solving yields:
$\frac{18}{x^3} = 0$ (multiply by x^3)
$\mathbf{x^3} \cdot (\frac{18}{x^3}) = \mathbf{x^3} \cdot 0$
18 = 0, no solution.
Thus, f has no inflection points. Hence, we mark only the value where f is undefined on the number line and determine the concavity of f:



Pick – 1 | Pick 1
f "(– 1) = – 18 | f "(1) = 18

Thus, f is concave up on $(0, \infty)$ and concave down on $(-\infty, 0)$.

VIII) Now, we will sketch the graph:



Ex. 20 $f(x) = \frac{10}{x^2+3}$.

<u>Solution:</u>

I) Since $x^2 + 3 \neq 0$, then the domain of f is $(-\infty, \infty)$.

II) Since $f(0) = \frac{10}{(0)^2+3} = \frac{10}{3}$, the y-intercept is $(0, \frac{10}{3})$.

Setting f(x) = 0 and solving yields:

$\frac{10}{x^2+3} = 0$ (multiply by $x^2 + 3$)

$(x^2 + 3)\bullet \frac{10}{x^2+3} = 0\bullet(x^2 + 3)$

10 = 0, no solution.
So, there are no x-intercepts.

III) $f'(x) = \frac{d}{dx}[\frac{10}{x^2+3}] = \frac{d}{dx}[10(x^2 + 3)^{-1}]$

$= -10(x^2 + 3)^{-2}\bullet\frac{d}{dx}[x^2 + 3] = -10(x^2 + 3)^{-2}\bullet[2x]$

$= -20x(x^2 + 3)^{-2} = \frac{-20x}{(x^2+3)^2}$.

$f''(x) = \frac{d}{dx}[\frac{-20x}{(x^2+3)^2}]$

$= \frac{(x^2+3)^2\bullet\frac{d}{dx}[-20x] - (-20x)\bullet\frac{d}{dx}[(x^2+3)^2]}{(x^2+3)^4}$

$= \frac{(x^2+3)^2\bullet[-20] - (-20x)\bullet[2(x^2+3)]\bullet\frac{d}{dx}(x^2+3)}{(x^2+3)^4}$

$= \frac{(x^2+3)^2\bullet[-20] + 40x\bullet(x^2+3)\bullet(2x)}{(x^2+3)^4}$

$= \frac{(x^2+3)[(x^2+3)\bullet[-20] + 40x(2x)]}{(x^2+3)^4}$

$= \frac{(x^2+3)[-20x^2-60+80x^2]}{(x^2+3)^4}$

$= \frac{[60x^2-60]}{(x^2+3)^3} = \frac{60(x^2-1)}{(x^2+3)^3}$.

IV) The function has no vertical asymptotes. But,

$$\lim_{x\to\pm\infty}\frac{10}{x^2+3} = \lim_{x\to\pm\infty}\frac{\frac{10}{x^2}}{\frac{x^2}{x^2}+\frac{3}{x^2}} = \lim_{x\to\pm\infty}\frac{\frac{10}{x^2}}{1+\frac{3}{x^2}} = \frac{0}{1+0} = 0.$$

Thus, f has a horizontal asymptote of y = 0.

V) Since $x^2 + 3 \neq 0$, f '(x) is defined for all real numbers. Setting f '(x) = 0 and solving yields:

$\frac{-20x}{(x^2+3)^2} = 0$ (multiply by $(x^2 + 3)^2$)

$\mathbf{(x^2 + 3)^2} \cdot \frac{-20x}{(x^2+3)^2} = 0 \cdot \mathbf{(x^2 + 3)^2}$

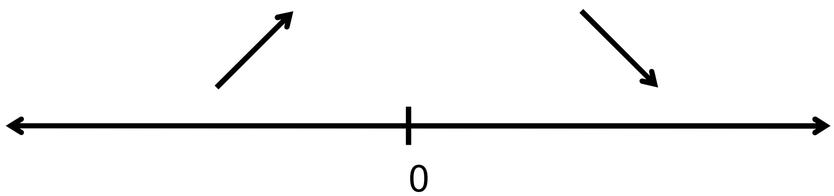
$-20x = 0$

$x = 0$

Evaluating f at this value yields:

$f(0) = \frac{10}{(0)^2+3} = \frac{10}{3}$.

So, $(0, \frac{10}{3})$ is the critical point. Marking x = 0 on the number line, we can find where f is increasing and decreasing:



Pick – 1 | Pick 1
$f'(-1) = 1.25$ | $f'(1) = -1.25$

Thus, f is increasing $(-\infty, 0)$ and decreasing on $(0, \infty)$.

VI) $f''(0) = \frac{60((0)^2-1)}{((0)^2+3)^3} = \frac{-60}{27} = -\frac{20}{9}$ so $(0, \frac{10}{3})$ is a relative maximum (verifies the results in part V).

VII) Since $x^2 + 3 \neq 0$, then f "(x) is defined for all real numbers. Setting f "(x) = 0 and solving yields:

$\frac{60(x^2-1)}{(x^2+3)^3} = 0$ (multiply by $(x^2 + 3)^3$)

$\mathbf{(x^2 + 3)^3} \cdot \frac{60(x^2-1)}{(x^2+3)^3} = \mathbf{(x^2 + 3)^3} \cdot 0$

$60(x^2 - 1) = 0$

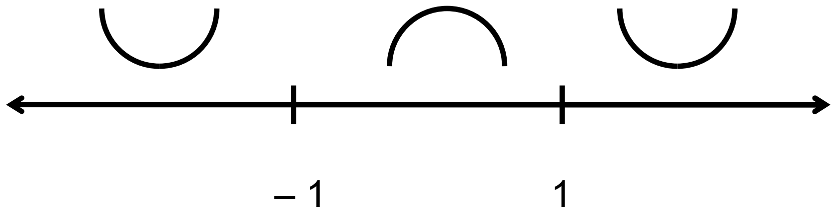
$60(x - 1)(x + 1) = 0$

$x = 1$ or $x = -1$.

Evaluating f at these values yields:

$f(-1) = 2.5$, and $f(1) = 2.5$.

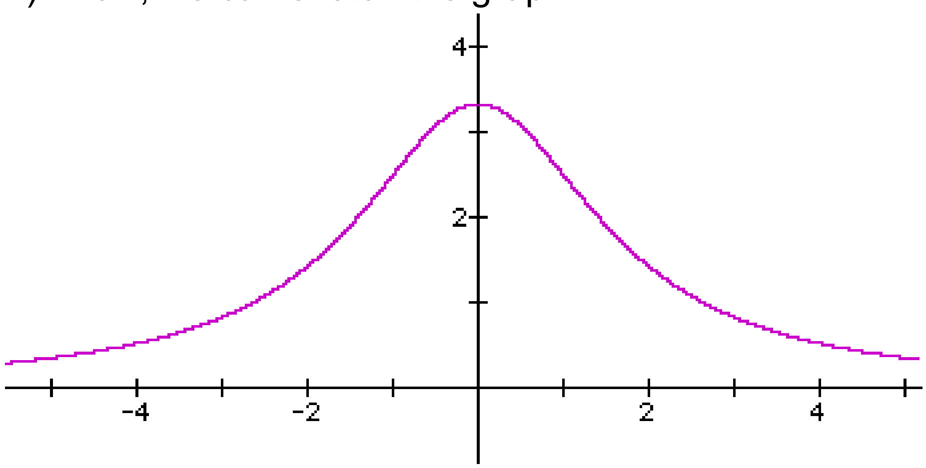
Thus, $(-1, 2.5)$ and $(1, 2.5)$ are the possible inflection points. Marking these values on the number line, we can find the concavity of f:



Pick – 2	Pick 0	Pick 2
f "(– 2)	f "(0)	f "(2)
≈ 0.525	≈ – 2.222	≈ 0.525

Hence, f is concave up on (– ∞, – 1) U (1, ∞) and concave down on (– 1, 1).

VIII) Now, we can sketch the graph:



Ex. 21 $f(x) = \frac{2x^2}{x^2 - 16}$.

Solution:

I) Since $x^2 - 16 = (x - 4)(x + 4)$, then f is undefined at x = 4 and – 4. Thus, the domain of f is (– ∞, – 4) U (– 4, 4) U (4, ∞).

II) Since $f(0) = \frac{2(0)^2}{(0)^2 - 16} = \frac{0}{-16} = 0$, the y-intercept is (0, 0).

Setting f(x) = 0 and solving yields:

$\frac{2x^2}{x^2 - 16} = 0$ (multiply by $x^2 - 16$)

$(x^2 - 16)\bullet \frac{2x^2}{x^2 - 16} = 0\bullet(x^2 - 16)$

$2x^2 = 0$
$x = 0$
So, the x-intercept is (0, 0).

III) $f'(x) = \frac{d}{dx}[\frac{2x^2}{x^2-16}] = \frac{(x^2-16)\bullet\frac{d}{dx}[2x^2] - 2x^2\bullet\frac{d}{dx}[x^2-16]}{(x^2-16)^2}$

$= \frac{(x^2-16)\bullet[4x] - 2x^2\bullet[2x]}{(x^2-16)^2} = \frac{4x^3 - 64x - 4x^3}{(x^2-16)^2} = \frac{-64x}{(x^2-16)^2}$

$f''(x) = \frac{d}{dx}[\frac{-64x}{(x^2-16)^2}]$

$= \frac{(x^2-16)^2\bullet\frac{d}{dx}[-64x] - (-64x)\bullet\frac{d}{dx}[(x^2-16)^2]}{(x^2-16)^4}$

$= \frac{(x^2-16)^2\bullet[-64] - (-64x)\bullet[2(x^2-16)]\bullet\frac{d}{dx}(x^2-16)}{(x^2-16)^4}$

$= \frac{(x^2-16)^2\bullet[-64] + 128x\bullet(x^2-16)\bullet(2x)}{(x^2-16)^4}$

$= \frac{(x^2-16)[(x^2-16)\bullet[-64] + 128x(2x)]}{(x^2-16)^4}$

$= \frac{(x^2-16)[-64x^2 + 1024 + 256x^2]}{(x^2-16)^4}$

$= \frac{[192x^2 - 1024]}{(x^2-16)^3} = \frac{64(3x^2+16)}{(x^2-16)^3}$.

IV) Since $\lim_{x\to 4^{\pm}} \frac{2x^2}{x^2-16} = \pm\infty$ and $\lim_{x\to -4^{\pm}} \frac{2x^2}{x^2-16} = \pm\infty$, f has two vertical asymptotes at x = – 4 and x = 4.

Also, $\lim_{x\to\pm\infty} \frac{2x^2}{x^2-16} = \lim_{x\to\pm\infty} \frac{\frac{2x^2}{x^2}}{\frac{x^2}{x^2} - \frac{16}{x^2}} = \lim_{x\to\pm\infty} \frac{2}{1-\frac{16}{x^2}}$

$= \frac{2}{1-0} = \frac{2}{1} = 2$. Thus, f has a horizontal asymptote of y = 2.

V) f '(x) is undefined at x = ± 4, but x = ± 4 are not in the domain of f. Thus, these are not critical values.

Setting f '(x) = 0 and solving yields:

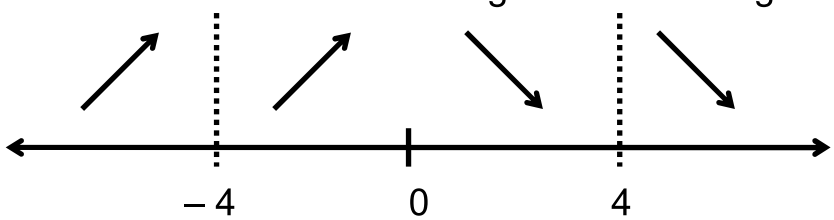
$\frac{-64x}{(x^2-16)^2} = 0$ (multiply by $(x^2 - 16)^2$)

$\mathbf{(x^2 - 16)^2} \bullet \frac{-64x}{(x^2-16)^2} = 0 \bullet \mathbf{(x^2 - 16)^2}$

$-64x = 0$
$x = 0$
Evaluating f at this value yields:

$f(0) = \frac{2(0)^2}{(0)^2-16} = \frac{0}{-16} = 0,$

So, (0, 0) is the critical point. Marking x = 0 and the values where f is undefined on the number line, we can find where f is increasing and decreasing:



Pick – 5	Pick – 2	Pick 2	Pick 5
f '(– 5)	f '(– 2)	f '(2)	f '(5)
≈ 3.95	≈ 0.89	≈ – 0.89	≈ – 3.95

Thus, f is increasing $(-\infty, -4) \cup (-4, 0)$ and decreasing on $(0, 4) \cup (4, \infty)$.

VI) $f''(0) = \frac{64(3(0)^2+16)}{((0)^2-16)^3} = \frac{1024}{-4096} = -\frac{1}{4}$ so (0, 0) is a relative maximum (verifies the results in part V).

VII) f '(x) is undefined at x = ± 4, but x = ± 4 are not in the domain of f. Thus, these values are not to be considered. Setting f '(x) = 0 and solving yields:

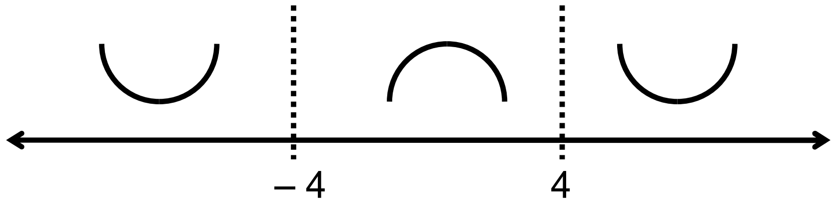
$\frac{64(3x^2+16)}{(x^2-16)^3} = 0$ (multiply by $(x^2 - 16)^3$)

$\mathbf{(x^2 - 16)^3} \bullet \frac{64(3x^2+16)}{(x^2-16)^3} = \mathbf{(x^2 - 16)^3} \bullet 0$

$64(3x^2 + 16) = 0$
$3x^2 + 16 = 0$

$x = \pm \sqrt{\frac{-16}{3}}$, no solution.

Thus, there are no inflection points. Marking only values that make f undefined on the number line, we can find the concavity of f:



Pick – 5	Pick 0	Pick 5
f "(– 5)	f "(0)	f "(5)
≈ 7.99	= – 0.25	≈ 7.99

Hence, f is concave up on $(-\infty, -4) \cup (4, \infty)$ and concave down on $(-4, 4)$.

VIII) Now, we can sketch the graph: