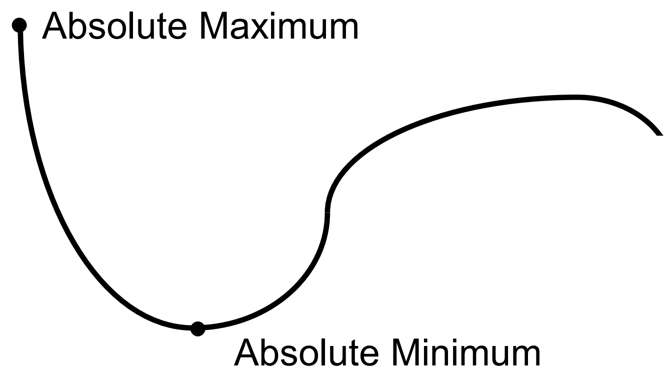
Sect 3.4 and 3.5 – Optimization Problems

Before we can optimize applications, we need to discuss what is meant by an absolute maximum and minimum. These ideas are different from relative maximum and relative minimum.

Absolute Extrema:
Let f be a function defined on an interval I that contains the point c. Then

a) f(c) is an <u>absolute maximum</u> of f if f(c) ≥ f(x) for all x in I.

b) f(c) is an <u>absolute minimum</u> of f if f(c) ≤ f(x) for all x in I.



If f is continuous function on [a, b], then the absolute maximum and minimum will occur at either a, b, or any critical points c_1, c_2, c_3, … This suggests a procedure for finding the absolute extrema of a function:

<u>Procedure</u>

1) Check to see if f is continuous on [a, b].

2) Find the critical values c_1, c_2, … where $f'(c_i) = 0$ or is undefined. Note: to be critical values, c_1, c_2, … must be in [a, b].

3) Find f (a), f (b), f (c_1), f (c_2), … The largest is the absolute maximum and the smallest is the absolute minimum.

Ex. 1 Find the absolute extrema of $f(x) = x^5 - 5x^4 + 1$ on the interval $[-3, 2]$.

Solution

1) Since f is a polynomial, it is continuous on $[-3, 2]$.

2) $f'(x) = \frac{d}{dx}[x^5 - 5x^4 + 1] = 5x^4 - 20x^3$.
Since f ' is a polynomial, it is defined for all real numbers. Setting $f'(x) = 0$ and solving yields:
$5x^4 - 20x^3 = 0$
$5x^3(x - 4) = 0$
$x = 0$ or $x = 4$, but $x = 4$ is not in $[-3, 2]$.
Therefore, $x = 0$ is the only critical value.

3) Evaluating f at $x = -3$, 0, and 2 yields:
$f(-3) = (-3)^5 - 5(-3)^4 + 1 = -243 - 405 + 1$
$= -647$
$f(0) = (0)^5 - 5(0)^4 + 1 = 1$
$f(2) = (2)^5 - 5(2)^4 + 1 = -47$

Thus, f has an absolute maximum of 1 at $x = 0$ and an absolute minimum of -647 at $x = -3$.

Ex. 2 Find the absolute extrema of $f(x) = x^3 - 12x$ on the interval $[-3, 3]$.

Solution

1) Since f is a polynomial, it is continuous on $[-3, 3]$.

2) $f'(x) = \frac{d}{dx}[x^3 - 12x] = 3x^2 - 12$.
Since f ' is a polynomial, it is defined for all real numbers. Setting $f'(x) = 0$ and solving yields:
$3x^2 - 12 = 0$
$3(x^2 - 4) = 0$
$3(x - 2)(x + 2) = 0$
$x = -2$ and $x = 2$ both of which are in $[-3, 3]$.
Therefore, $x = -2$ and 2 are the critical values.

3) Evaluating f at $x = -3$, -2, 2, and 3 yields:
$f(-3) = (-3)^3 - 12(-3) = -27 + 36 = 9$

$f(-2) = (-2)^3 - 12(-2) = -8 + 24 = 16$
$f(2) = (2)^3 - 12(2) = 8 - 24 = -16$
$f(3) = (3)^3 - 12(3) = 27 - 36 = -9$

Thus, f has an absolute maximum of 16 at x = – 2 and an absolute minimum of – 16 at x = 2.

One Absolute Extreme

If f is continuous on an interval I and x = c is the only critical value, then

1) If f "(c) > 0, the f(c) is an absolute minimum ($\cup$).
2) If f "(c) < 0, the f(c) is an absolute maximum ($\cap$).

Ex. 3 Find the absolute extrema of $f(x) = -0.5x^2 + 10x - 37$ on the interval $(0, \infty)$.

Solution

1) Since f is a polynomial, it is continuous on $(0, \infty)$.

2) $f'(x) = \frac{d}{dx}[-0.5x^2 + 10x - 37] = -x + 10$.
Since f ' is a polynomial, it is defined for all real numbers. Setting f '(x) = 0 and solving yields:
$-x + 10 = 0$
$x = 10$ which is in $(0, \infty)$. Therefore, x = 10 is the only critical value.

3) $f''(x) = \frac{d}{dx}[-x + 10] = -1$, so $f''(10) = -1 < 0$ ($\cap$ – absolute maximum). Evaluate f at x = 10 yields:
$f(10) = -0.5(10)^2 + 10(10) - 37 = -50 + 100 - 37 = 13$.

So, f has an absolute maximum of 13 at x = 10 and f has no absolute minimum.

Ex. 4 Find the absolute extrema of $f(x) = -4x^3 - 6x^2 + 24x$ on the interval $(-\infty, 0)$.

Solution

1) Since f is a polynomial, it is continuous on $(-\infty, 0)$.

2) $f'(x) = \frac{d}{dx}[-4x^3 - 6x^2 + 24x] = -12x^2 - 12x + 24$.

Since f ' is a polynomial, it is defined for all real numbers. Setting f '(x) = 0 and solving yields:

$-12x^2 - 12x + 24 = 0$
$-12(x^2 + x - 2) = 0$
$-12(x - 1)(x + 2) = 0$
$x = 1$ and $x = -2$

But, only x = – 2 which is in $(-\infty, 0)$. Therefore, x = – 2 is the only critical value.

3) $f''(x) = \frac{d}{dx}[-12x^2 - 12x + 24] = -24x - 12$, so
$f''(-2) = 48 - 12 = 36 > 0$ ($\cup$ – absolute minimum). Evaluate f at x = – 2 yields:
$f(-2) = -4(-2)^3 - 6(-2)^2 + 24(-2) = 32 - 24 - 48$
$= -56$.
So, f has an absolute minimum of – 56 at x = – 2 and no absolute maximum.

Ex. 5 A radio station conducted survey on finding the percentage of people tuned into their station x hours after 5 pm. They found that this percentage can be modeled by $f(x) = \frac{1}{8}(-2x^3 + 27x^2 - 108x + 240)$.

a) At what time between 5 pm and midnight are the most people listening to the station? What is that percentage?

b) At what time between 5 pm and midnight are the fewest people listening to the station? What is that percentage?

<u>Solution:</u>

1) Since 5 pm corresponds to x = 0 and midnight corresponds to x = 7, the domain of f is [0, 7]. Thus, $f(x) = \frac{1}{8}(-2x^3 + 27x^2 - 108x + 240)$ is continuous on [0, 7].

2) $f'(x) = \frac{d}{dx}[\frac{1}{8}(-2x^3 + 27x^2 - 108x + 240)]$
$= \frac{1}{8}(-6x^2 + 54x - 108)$.

Since f ' is a polynomial, it is defined for all real numbers. Setting f '(x) = 0 and solving yields:

$\frac{1}{8}(-6x^2 + 54x - 108) = 0$

$-\frac{6}{8}(x^2 - 9x + 18) = 0$

$-\frac{3}{4}(x - 6)(x - 3) = 0$

x = 3 and x = 6 which are both critical values.

3) $f(0) = \frac{1}{8}(-2(0)^3 + 27(0)^2 - 108(0) + 240) = 30\%$

$f(3) = \frac{1}{8}(-2(3)^3 + 27(3)^2 - 108(3) + 240) = 13.125\%$

$f(6) = \frac{1}{8}(-2(6)^3 + 27(6)^2 - 108(6) + 240) = 16.5\%$

$f(7) = \frac{1}{8}(-2(7)^3 + 27(7)^2 - 108(7) + 240) = 15.125\%$

a) The highest percentage of people listening to the station is 30% at 5 p.m.

b) The lowest percentage of people listening to the station is 13.125% at 8 p.m.

Ex. 6 A bookstore can obtain a certain gift book from the publisher at a cost of $3 per book. The bookstore has been selling 200 copies of the book per month at $15 per copy. The bookstore estimates that for each $1 reduction in price, they would be able to sell 20 more books. At what price should the bookstore sell the book to generate the greatest possible profit?

<u>Solution:</u>

Let x = price per book

Let y = the number of books sold

If the price of the book is x_1 = $15, then the number of books sold is $y_1 = 200$. If the price of the book is x_2 = $14, then the number of books sold is $y_2 = 220$. We can use the formula for the slope to calculate the rate of change:

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{220 - 200}{14 - 15} = \frac{20}{-1} = -20.$$

Using the point-slope formula, we can find y as a function of x:

$y - y_1 = m(x - x_1)$

$y - 200 = -20(x - 15)$

$y - 200 = -20x + 300$
$y = -20x + 500.$

Now, we can find the total revenue function, R(x):
R(x) = (the price per book)(the number of books sold)
$= xy = x(-20x + 500) = -20x^2 + 500x.$
We also need to find the total cost function:
C(x) = (the cost per book) (the number of books sold)
$= 3(-20x + 500) = -60x + 1500.$
The total profit is revenue minus cost:
$P(x) = R(x) - C(x) = -20x^2 + 500x - (-60x + 1500)$
$= -20x^2 + 560x - 1500$. The domain of this function is $(0, \infty)$ since x has to be greater than zero. We can now go through our steps to optimize the profit:

1) P(x) is a polynomial so it is continuous on $(0, \infty)$.

2) $P'(x) = \frac{d}{dx}[-20x^2 + 560x - 1500] = -40x + 560.$
Since P ' is a polynomial, it is defined for all real numbers. Setting P '(x) = 0 and solving yields:
$-40x + 560 = 0$
$x = 14$ which is in $(0, \infty)$. Therefore, x = 14 is the only critical value.

3) $P''(x) = \frac{d}{dx}[-40x + 560] = -40,$
so $P''(14) = -40 < 0$ ($\cap$ – absolute maximum)
$P(14) = -20(14)^2 + 560(14) - 1500$
$= -3920 + 7840 - 1500 = 2420.$
So, the profit is maximized at \$2420 when the price is \$14 per book.

Ex. 7 A farmer can get \$2 per bushel for their potatoes on July 1st. After that, the price drops by 2 cents per bushel per day. A farmer has 80 bushels in the field on July 1st and estimates that the crop is increasing at a rate of one bushel per day. When should the farmer harvest the potatoes to maximize revenue?

<u>Solution:</u>

Let x = the number of days after July 1st
Since the price is decreasing by 2 cents per day, then after x days, the price is $2 - 0.02x$. Since the number of

bushels is increasing by one bushel per day, the number of bushels after x days is 80 + x. We can then find the revenue function: R(x) = (price)(number of bushels) $= (2 - 0.02x)(80 + x) = 160 + 0.4x - 0.02x^2$. The domain of the function is $[0, \infty)$ since x could equal 0. We can proceed to optimize the total revenue function:

1) Since R is a polynomial, it is continuous on $[0, \infty)$.

2) $R'(x) = \frac{d}{dx}[160 + 0.4x - 0.02x^2] = 0.4 - 0.04x$.
Since R ' is a polynomial, it is defined for all real numbers. Setting R '(x) = 0 and solving yields:
$0.4 - 0.04x = 0$
x = 10 which is in $[0, \infty)$. Therefore, x = 10 is the only critical value.

3) $R''(x) = \frac{d}{dx}[0.4 - 0.04x] = -0.04$, so $R''(10)$
$= -0.04 < 0$ ($\cap$ – absolute maximum)
$R(10) = 160 + 0.4(10) - 0.02(10)^2 = 160 + 4 - 2$
$= 162$.

Since x = 10 corresponds to ten days after July 1^{st}, the farmer should harvest on July 11 to maximize the total revenue.

Ex. 8 A cylindrical can is to hold 4π cubic inches of frozen orange juice. The cost per square inch of constructing a metal top and bottom is twice the cost per square inch of constructing the cardboard side. What are the dimensions of the least expensive can?

Solution:

The volume of the cylinder is $V = \pi r^2 h = 4\pi$. Solving for h yields: $h = \frac{4\pi}{\pi r^2} = \frac{4}{r^2}$

The surface area of the top and bottom is $2\pi r^2$ and for the sides is $2\pi rh$. Since the top and bottom cost twice as much as the sides, the total cost = $\mathbf{2}(2\pi r^2) + 2\pi rh$.

Substituting, $h = \frac{4}{r^2}$, we get:

$$C(r) = 4\pi r^2 + 2\pi r\left(\frac{4}{r^2}\right) = 4\pi r^2 + 8\pi r^{-1} = 4\pi r^2 + \frac{8\pi}{r}.$$

The domain of C is $(0, \infty)$. We can now proceed to optimize this problem.

1) $C(r)$ is continuous on $(0, \infty)$.

2) $C'(r) = \frac{d}{dr}[4\pi r^2 + 8\pi r^{-1}] = 8\pi r - 8\pi r^{-2} = 8\pi r - \frac{8\pi}{r^2}$.
C '(r) is defined on $(0, \infty)$. Setting C '(r) = 0 and solving yields:
$8\pi r - \frac{8\pi}{r^2} = 0$ (multiply by r^2)
$8\pi r^3 - 8\pi = 0$
$8\pi(r^3 - 1) = 0$
$8\pi(r - 1)(r^2 + r + 1) = 0$
$r = 1$ and $r^2 + r + 1 \neq 0$. Thus, $r = 1$ is the only critical value and $h = \frac{4}{r^2} = 4$.

3) $C''(r) = \frac{d}{dr}[8\pi r - 8\pi r^{-2}] = 8\pi + 16\pi r^{-3} = 8\pi + \frac{16\pi}{r^3}$.
$C''(1) = 8\pi + \frac{16\pi}{(1)^3} = 24\pi > 0$
($\cup$ – Absolute Minimum)

The cost will be minimized when the radius is 1 in and the height is 4 in.

<u>Minimizing Total Inventory Costs (T.I.C.)</u>

Ex. 9 An electronics firm uses 600 cases of transistors each year. The cost of storing one case for one year is 90 cents, and there is an order fee of $30 per shipment. Also, it cost the firm $3 per case ordered. How many cases should the firm order each time to keep the total cost at a minimum? (Assume that the transistors are used at a constant rate throughout the year and each shipment arrives just as the preceding one is used up.)

<u>Solution:</u>
The total inventory cost consists of three parts: the storage cost, the order cost, and the purchase cost.

T.C.I.(x) = Storage Cost + Order Cost + Purchase Cost

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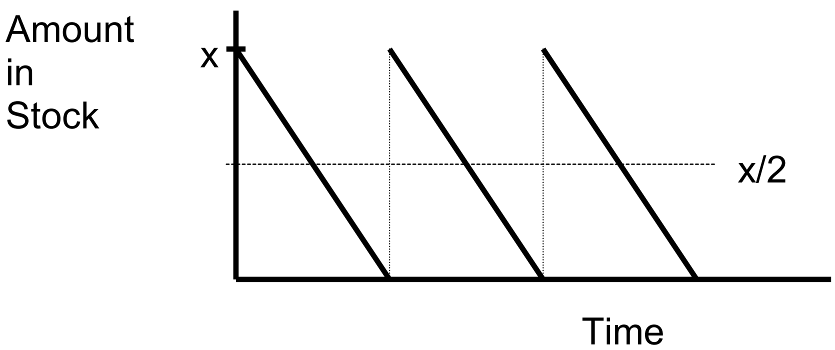
$$= \begin{matrix}\text{Storage Cost} \\ \text{of 1 item for} \\ \text{1 year}\end{matrix} \bullet \begin{matrix}\text{Average \#} \\ \text{of items} \\ \text{in stock}\end{matrix} + \begin{matrix}\text{Ordering} \\ \text{Cost per} \\ \text{Shipment}\end{matrix} \bullet \begin{matrix}\text{Number} \\ \text{of} \\ \text{Shipments}\end{matrix} + \begin{matrix}\text{Total Items} \\ \text{purchased} \\ \text{per year}\end{matrix} \bullet \begin{matrix}\text{Cost} \\ \text{per} \\ \text{item}\end{matrix}$$

Let x = the number of items ordered per shipment. We are to assume that the transistors are being used at a constant rate and each shipment arrives just as the existing stock is used up. We can make a graph of the firm's inventory:



The average number of items in stock $= \frac{x}{2}$. The number of shipments the firm receives per year is:

$$\frac{\text{Total Items purchased per year}}{x} = \text{Number of shipments}$$

Thus, T.C.I.(x) is:

$$= \begin{matrix}\text{Storage Cost} \\ \text{of 1 item for} \\ \text{1 year}\end{matrix} \bullet \frac{x}{2} + \begin{matrix}\text{Ordering} \\ \text{Cost per} \\ \text{Shipment}\end{matrix} \bullet \frac{\text{Total Items purchased per year}}{x} + \begin{matrix}\text{Total Items} \\ \text{purchased} \\ \text{per year}\end{matrix} \bullet \begin{matrix}\text{Cost} \\ \text{per} \\ \text{item}\end{matrix}$$

Plugging in, we get:

$$\text{T.I.C (x)} = (0.9)\frac{x}{2} + (30)\frac{600}{x} + (600)(3)$$

$= 0.45x + \frac{18000}{x} + 1800 = 0.45x + 18000x^{-1} + 1800$. The domain of the function is $(0, \infty)$. We can now optimize this function:

1) T.I.C (x) is continuous on $(0, \infty)$.

2) $\text{T.I.C}'(x) = \frac{d}{dx}[0.45x + 18000x^{-1} + 1800]$

$= 0.45 - 18000x^{-2} = 0.45 - \frac{18000}{x^2}$.

T.I.C ' is defined on $(0, \infty)$. Setting $\text{T.I.C}'(x) = 0$ and solving yields:

$0.45 - \frac{18000}{x^2} = 0$ (multiply by x^2)

$0.45x^2 - 18000 = 0$

$x^2 = 40{,}000$

$x = \pm 200$, but $x = 200$ is in $(0, \infty)$, so $x = 200$ is the only critical value.

3) $\text{T.I.C}''(x) = \frac{d}{dx}[0.45 - 18000x^{-2}] = 36000x^{-3}$

$= \frac{36000}{x^3}$. Hence, $\text{T.I.C}''(200) = \frac{36000}{(200)^3} > 0$

($\cup$ – Absolute Minimum)

So, they should order 200 cases at a time to minimize the total inventory costs. They will have to place an order 3 times a year.

<u>Elasticity of Demand</u> – measures the sensitivity of demand x(p) to changes in the price p.

$$\varepsilon(p) = -\frac{\%\text{ change in } x(p)}{\%\text{ change in } p} = -\frac{\dfrac{100\left(\frac{dx}{dp}\right)}{x(p)}}{\dfrac{100\left(\frac{dp}{dp}\right)}{p}} = -\frac{p}{x(p)} \bullet \frac{dx}{dp}.$$

If $\varepsilon(p) > 1$, the demand is said to be <u>elastic</u> (A small % change in price makes for a larger % change in demand). The total revenue is decreasing. (Lower prices)

If $\varepsilon(p) < 1$, the demand is said to be <u>inelastic</u> (A small % change in price makes for a smaller % change in demand). The total revenue is increasing. (Raise prices)

If ε (p) = 1, the demand is said to have a unit of elasticity (A small % change in price makes for the same % change in demand). The total revenue is maximized.

Ex. 10 Suppose that x(p) = 500 – 2p units of a commodity are demanded when p dollars per unit are charged where $0 \le p \le 250$.

a) Determine where the demand is elastic, inelastic, and of unit of elasticity with respect to price.

b) Use the results from part a to determine where the revenue function is increasing and decreasing and the price at which the revenue is maximized.

Solution:

Since x(p) = 500 – 2p, then $\frac{dx}{dp} = -2$. Thus,

$$\varepsilon(p) = -\frac{p}{x(p)} \bullet \frac{dx}{dp} = -\frac{p}{500-2p}(-2) = \frac{2p}{500-2p} = \frac{p}{250-p}$$

a) Setting ε (p) = 1 and solving, we find that:

$\frac{p}{250-p} = 1$ (multiply by 250 – p)

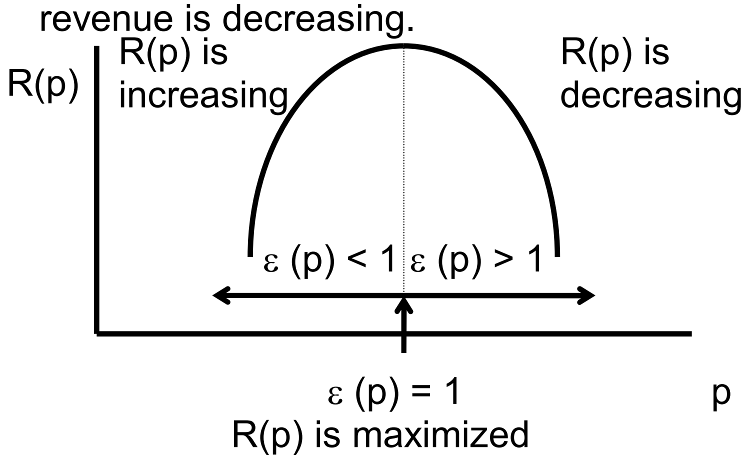
p = 250 – p

2p = 250

p = \$125

Thus, we have a unit of elasticity at p = \$125. So the total revenue is maximized at p = \$125.

If p < 125, then ε (p) < 1, so the demand is inelastic. The revenue is increasing. If p > 125, then ε (p) > 1, so the demand is elastic. The revenue is decreasing.



b) R is increasing on [0, 125) and decreasing on (125, 250].

Let's now find the total revenue function explicitly and verify our results:

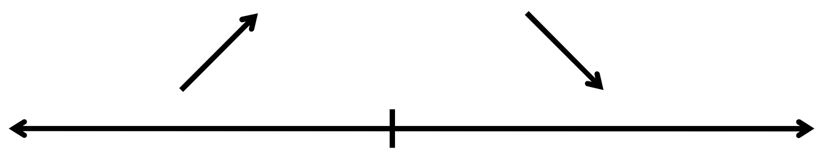
$R(p) = p \bullet x(p) = (500 - 2p) = 500p - 2p^2$

1) R(p) is a polynomials so it is continuous on [0, 250].

2) $R'(p) = 500 - 4p$ which is defined in [0, 250]. Setting $R'(p) = 0$ and solving yields:

$$500 - 4p = 0$$
$$4p = 500$$
$$p = 125$$

Marking p = 125 on the number line, we can determine where R is increasing and decreasing:



125

Pick 100	Pick 200
$R'(100) = 100$	$R'(200) = -300$

Thus, R is increasing on [0, 125) and decreasing on (125, 250].

3) $R''(p) = -4$, so $R''(125) = -4 < 0$ ($\cap$ – Absolute Maximum). Thus the total revenue is maximized at p = \$125.