Section 3.6 – Marginal Analysis

Marginal Analysis is the study of how quantities like cost, revenue, and profit change with an increase in production. Typically, the increase in production is an increase of one unit, so in essence, we will examine how various quantities are affected when the next unit is produced.

Recall that the following are equivalent:

$$f'(x) = \frac{dy}{dx} = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}.$$

As Δx gets smaller and smaller, $\frac{\Delta y}{\Delta x}$ gets close to $f'(x)$.

Thus for very small Δx, $\frac{\Delta y}{\Delta x} \approx f'(x)$. Solving for Δy yields:

$$\Delta y \approx f'(x)\bullet\Delta x$$

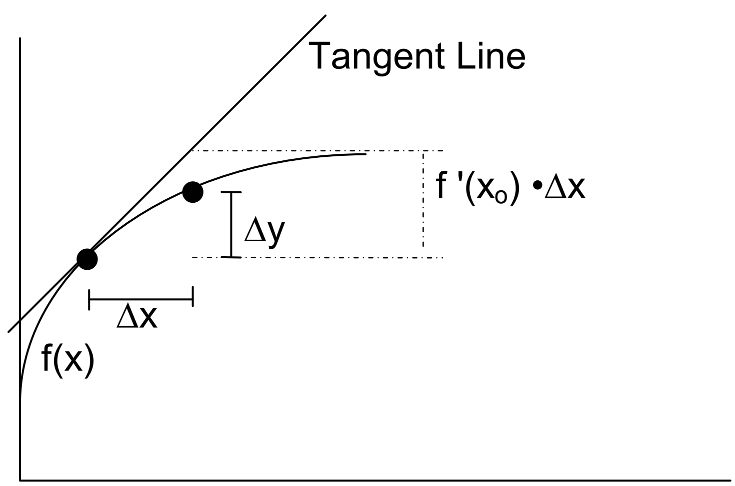
If we are examining the behavior of the function at the point $(x_o, f(x_o))$, then $\Delta y \approx f'(x_o)\bullet\Delta x$.

Approximation by Increments:
If $f(x)$ is differentiable at $x = x_o$ and Δx is a small change in x, then

$$\Delta y \approx f'(x_o)\bullet\Delta x$$

and $f(x_o) + \Delta y = f(x_o + \Delta x) \approx f(x_o) + f'(x)\bullet\Delta x$

Remember that Δy is the actual change in the function and $f'(x_o)\bullet\Delta x$ is the approximation or the change that occurs on the tangent line:



Ex. 1 Estimate how much the function $f(x) = 2x^2 - 8x + 5$ will change as x increases from 7 to 7.3.

<u>Solution:</u>

First, $\Delta x = 7.3 - 7 = 0.3$

Now, computing the derivative, we get:

$$f'(x) = \frac{d}{dx}[2x^2 - 8x + 5] = 4x - 8.$$

Thus, $\Delta y \approx f'(7)\bullet(0.3) = (4(7) - 8)\bullet(0.3) = (20)(0.3) = 6.$

We estimate that the function value will increase by 6.

Ex. 2 Estimate how much $g(x) = x^2 - x^3$ will change as

a) x decreases from 3 to 2.8.

b) x increases from 1 to 1.1.

<u>Solution:</u>

a) First, $\Delta x = 2.8 - 3 = -0.2$

Now, computing the derivative, we get:

$$g'(x) = \frac{d}{dx}[x^2 - x^3] = 2x - 3x^2.$$

Thus, $\Delta y \approx g'(3)\bullet(-0.2) = (2(3) - 3(3)^2)\bullet(-0.2)$

$= (-21)(-0.2) = 4.2.$

We estimate that the value will increase by 4.2.

b) First, $\Delta x = 1.1 - 1 = 0.1$

From part a, $g'(x) = 2x - 3x^2$.

Thus, $\Delta y \approx g'(1)\bullet(0.1) = (2(1) - 3(1)^2)\bullet(0.1)$

$= (-1)(0.1) = -0.1.$

We estimate that the value will decrease by 0.1.

This idea can also be extended to the discussion of percentage change. The percentage change in a function f ($\%\Delta f$) is the change in quantity divided by the original quantity times 100%.

Percentage Change in f $= 100\% \bullet \dfrac{\text{Change in f}}{\text{Size of f}} = 100\% \bullet \dfrac{\Delta f}{f(x)}$

But if Δx is a small change in x and since $\Delta f = \Delta y$, then $\Delta f \approx f'(x)\bullet\Delta x$. Thus,

$$\%\Delta f \approx \mathbf{100\%} \bullet \frac{f'(x) \bullet \Delta x}{f(x)}.$$

Do not confuse this with the percentage rate of change!

Ex. 3 Estimate the percentage change in $g(x) = x^2 - x^3$ as x decreases from 3 to 2.8.

<u>Solution:</u>

Recall from Ex 2a that $\Delta x = -0.2$ and $\Delta g = \Delta y \approx 4.2$.

Since $g(3) = (3)^2 - (3)^3 = 9 - 27 = -18$, then

$\%\Delta g \approx 100\% \bullet \frac{4.2}{-18} = -23\frac{1}{3}\%$.

Ex. 4 A manufacturer's total cost is $C(q) = \frac{1}{9}q^3 + 1037q + 400$ dollars when q units are produced. The current level of production is 3 units. Estimate the amount by which the manufacturer should decrease production to reduce the total cost by \$130.

<u>Solution:</u>

Here, we are given ΔC and we need to find Δq.

Computing the derivative, we get:

$$C'(q) = \frac{d}{dq}\left[\frac{1}{9}q^3 + 1037q + 400\right] = \frac{1}{3}q^2 + 1037.$$

Thus, $C'(3) = \frac{1}{3}(3)^2 + 1037 = 3 + 1037 = 1040$. Since $\Delta C = -130$, we can now solve for Δq:

$$\Delta C \approx C'(q) \bullet \Delta q$$

$$-130 \approx 1040\Delta q$$

$$-0.125 \approx \Delta q$$

The manufacturer should decrease production by about 0.125 units.

Ex. 5 At a certain factory, the daily output is $Q(K) = 750K^{1/2}$ units, where K denotes the firms capital investment. Estimate the percentage increase in capital investment that is needed to produce a 2.4% increase in output.

<u>Solution:</u>

Here, we are give $\%\Delta Q$ and we need to find ΔK. We will need to use the percentage change formula and solve for ΔK in terms of K. This will give us the percentage increase in capital investment.

Computing the derivative, we get:

$$Q'(K) = \frac{d}{dK}[750K^{1/2}] = 750 \bullet (1/2)K^{-1/2} = \frac{375}{\sqrt{K}}.$$

Substituting into the formula for the percentage change,

we get: $\%\Delta Q \approx \mathbf{100\%} \bullet \frac{Q'(K)\bullet\Delta K}{Q(K)}$

$$2.4\% \approx 100\%\bullet\frac{\frac{375}{\sqrt{K}}\bullet\Delta K}{750\sqrt{K}} \text{ (Invert \& Multiply)}$$

$$2.4\% \approx 100\%\bullet\frac{375}{\sqrt{K}}\bullet\Delta K \bullet \frac{1}{750\sqrt{K}} \text{ (Simplify)}$$

$$2.4\% \approx \frac{50\%\bullet\Delta K}{K} \text{ (Multiply both sides by K)}$$

$$2.4\%\bullet K \approx 50\%\bullet\Delta K \text{ (Solve for } \Delta K)$$

$$\Delta K \approx 0.048K$$

Thus, the capital investment will need to increase by about 4.8% in order for output to increase by 2.4%.

Ex. 6 Let $C(x) = 6x^2 - 5x + 3$ be the total cost function when x units are produced. Estimate ΔC when $x = 2$ & $\Delta x = 1$.

<u>Solution:</u>

Computing the derivative, we get:

$$C'(x) = \frac{d}{dx}[6x^2 - 5x + 3] = 12x - 5. \text{ Thus,}$$

$$\Delta C \approx C'(x)\bullet\Delta x = C'(2)(1) = (12(2) - 5)(1) = 19.$$

We estimate that when 2 units are produced, if the production level is increased by one unit, the total cost goes up \$19.

The actual cost to produce the 3rd can be found by computing $C(3) - C(2)$:

$$C(3) - C(2) = [6(3)^2 - 5(3) + 3] - [6(2)^2 - 5(2) + 3]$$
$$= [54 - 15 + 3] - [24 - 10 + 3] = [42] - [17] = \$25.$$

Notice that when $\Delta x = 1$, $\Delta C \approx C'(x)$. This is referred to as the **marginal cost** ("the cost to produce one more unit or the $(x + 1)^{st}$ unit). We can define the **marginal revenue** and the **marginal profit** in a similar fashion. Let's formalize this idea:

Marginal cost (the cost to produce one more) at $x = x_o$ is:
$MC(x_o) = C'(x_o) \approx C(x_o + 1) - C(x_o)$

Marginal Revenue (the amount received from selling one more) at $x = x_o$ is: $MR(x_o) = R'(x_o) \approx R(x_o + 1) - R(x_o)$

Marginal Profit (the profit received from producing and selling one more) at $x = x_o$ is: $MP(x_o) = P'(x_o) \approx P(x_o + 1) - P(x_o)$

Ex. 7 Let $C(x) = \frac{3}{5}x^2 + 5x + 102$ be the total cost function for producing x units of a particular commodity and $p(x) = -2x^2 + 3x + 62$ be the price at which all x units are sold.

a) Find the marginal revenue and the marginal cost.

b) Use the marginal cost to estimate the cost of producing the 4^{th} unit.

c) Find the actual cost of producing the fourth unit.

d) Use the marginal revenue to estimate the revenue derived from the sale of the 4^{th} unit.

e) Find the actual revenue derived from the sale of the 4^{th} unit.

f) If the average cost is $\frac{C(x)}{x}$, compute the average cost and the marginal average cost.

Solution:

a) The total revenue is x•p(x), so

$R(x) = x(-2x^2 + 3x + 62) = -2x^3 + 3x^2 + 62x.$

Computing the derivative, we get:

$MR(x) = \frac{d}{dx}[-2x^3 + 3x^2 + 62x] = -6x^2 + 6x + 62.$

Since $C(x) = \frac{3}{5}x^2 + 5x + 102$, then

$MC(x) = \frac{d}{dx}[\frac{3}{5}x^2 + 5x + 102] = \frac{6}{5}x + 5.$

b) To find the cost of producing the 4^{th} unit ($(x_o + 1)^{st}$ unit), we need to evaluate MC at x = 3 units (x_o):

$MC(3) = \frac{6}{5}(3) + 5 = 3.6 + 5 = \$8.60$

The cost to produce the 4^{th} unit is about \$8.60.

c) The actual cost for the 4^{th} unit is C(4) – C(3)

$= [\frac{3}{5}(4)^2 + 5(4) + 102] - [\frac{3}{5}(3)^2 + 5(3) + 102]$

$= [9.6 + 20 + 102] - [5.4 + 15 + 102]$

$= [131.6] - [122.4] = \$9.20.$

The actual cost to produce the 4^{th} unit is \$9.20.

d) To find the revenue of selling the 4^{th} unit ($(x_o + 1)^{st}$ unit), we need to evaluate MR at x = 3 units (x_o):

$MR(3) = -6(3)^2 + 6(3) + 62 = -54 + 18 + 62 = 26.$

The revenue from selling the 4^{th} unit is about \$26.

e) The actual revenue for selling the 4^{th} unit is
R(4) – R(3)
$= [-2(4)^3 + 3(4)^2 + 62(4)] - [-2(3)^3 + 3(3)^2 + 62(3)]$
$= [-128 + 48 + 248] - [-54 + 27 + 186]$
$= [168] - [159] = \$9.$
The actual revenue from selling the 4^{th} unit is $9.

f) We first find the average cost function:

$$AC = \frac{C(x)}{x} = \frac{\frac{3}{5}x^2 + 5x + 102}{x} = \frac{3}{5}x + 5 + \frac{102}{x}$$

Computing the derivative, we get:

$$MAC(x) = \frac{d}{dx}[\frac{3}{5}x + 5 + \frac{102}{x}] = \frac{d}{dx}[\frac{3}{5}x + 5 + 102x^{-1}]$$

$$= \frac{3}{5} - 102x^{-2} = \frac{3}{5} - \frac{102}{x^2}.$$

When average cost function is minimized, the marginal cost is equal to the average cost function. Recall that a function has a maximum or minimum value when its derivative is zero. We will differentiate the average cost function, set it equal to zero, and solve for MC(x). This will show that the marginal cost is equal to average cost. We will not show that the average cost is minimized as opposed to maximized at this point, but just assume it to be true for now.

$$AC\,'(x) = \frac{d}{dx}[\frac{C(x)}{x}] = \frac{x \bullet \frac{d}{dx}[C(x)] - C(x) \bullet \frac{d}{dx}[x]}{x^2} = \frac{x \bullet C'(x) - C(x) \bullet [1]}{x^2}$$

$$= \frac{x \bullet C'(x) - C(x)}{x^2}.$$

Setting the derivative equal to zero and solving for C '(x), we get:

$\frac{x \bullet C'(x) - C(x)}{x^2} = 0$ (multiply by x^2)

$x \bullet C'(x) - C(x) = 0$

$x \bullet C'(x) = C(x)$ (divide by x)

$C'(x) = \frac{C(x)}{x}$

But MC(x) = C '(x) and $AC(x) = \frac{C(x)}{x}$, thus MC(x) = AC(x). Thus, the average cost function is minimized when the marginal cost is equal to the average cost. This is an important relationship in economics.