Section 4.1 - Exponential Functions

Exponential functions are extremely important in both economics and science. It allows us to discuss the growth of money in a money market account as well as the growth of a population. In an exponential function, the growth rate of the function is depended upon the current size of the population. Exponential functions can also be used to model present value problems or how old an object is. To begin our discussion of exponential functions, let's look at an example drawn from Bennett and Brigg's Using and Understanding Mathematics, Addison Wesley Longman, Inc., © 1999, pg. 362 - 364:

Ex. 1 Suppose that a certain type of bacteria doubles in size every minute so long that there is enough nutrients to feed the colony. Also, suppose that one bacteria is placed in a bottle full of nutrients at 2 pm and at 3 pm, the bottle is full and all the bacteria starve to death.

a) When was the bottle half full?

b) Suppose at 2:56 pm, one of the bacteria tries to warn the colony that they will all die in four minutes. Would that bacteria be believed?

c) Suppose that at 2:59 pm, the colony realizes that they are in trouble and are able to find 4 more nutrient filled bottles and transport enough members of the colony so that they had an equal number of bacteria in each of the four bottles. How much longer will the colony survive?

Solution:

a) At 2 pm, there is one bacteria (2^0). Since, the population is doubling every minute, then at 2:01 pm, there are two bacteria (2^1). At 2:02 pm, there are four bacteria (2^2). At 2:03 pm, there are eight bacteria (2^3). This pattern will continue. When the bottle is full, there will be 2^{60}. One might guess that the bottle was half full at 2:30 pm, but that would be incorrect since the population at 2:31 pm will be double that at 2:30 pm. Let's make a chart:

Time	Number of Bacteria	Fraction of Bottle Full
2:00 pm	$1 = 2^0$	$1/2^{60}$ Full
2:01 pm	$2 = 2^1$	$2/2^{60} = 1/2^{59}$ Full
2:02 pm	$4 = 2^2$	$4/2^{60} = 1/2^{58}$ Full
• • •	• • •	• • •
2:58 pm	2^{58}	$2^{58}/2^{60} = 1/4$ Full
2:59 pm	2^{59}	$2^{59}/2^{60} = 1/2$ Full
3:00 pm	2^{60}	$2^{60}/2^{60} = 1$ Full

We can see that the bottle will be half full at 2:59 pm.

b) At 2:56 pm, one of the bacteria is yelling, "The End is Near!." At this point, there are 2^{56} bacteria in the colony. The bottle is $2^{56}/2^{60}$ full or $1/2^4 = 1/16$ full. It is unlikely that the bacteria will be believed.

c) If the colony is divided into four equal parts, then each new bottle is only 1/4 full. At 3:01 pm, each bottle is 1/2 full and at 3:02 pm, each bottle is full and the bacteria starves to death. Thus, the colony lasts only two more minutes.

In this last example, the population of the bacteria was growing exponentially. Notice that the base was not changing after each minute, but the exponent was. Thus, we can make the following definition:

<u>Exponential Function</u>

If $b > 0$ and $b \neq 1$, then $f(x) = b^x$ is an exponential function f with base b.

To see what an exponential function looks like, let's consider the following examples:

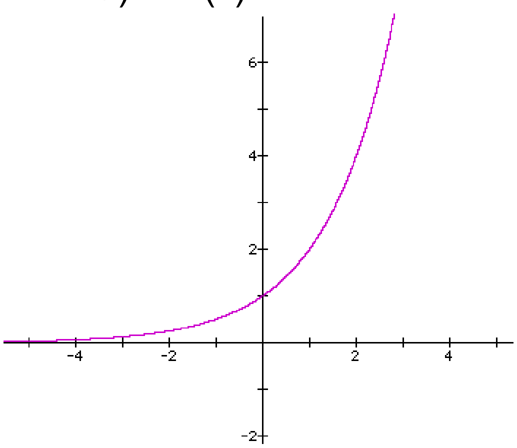
Ex. 2 Make a table of values and graph the following:

a) $f(x) = 2^x$ b) $g(x) = 3^x$ c) $h(x) = 4^x$

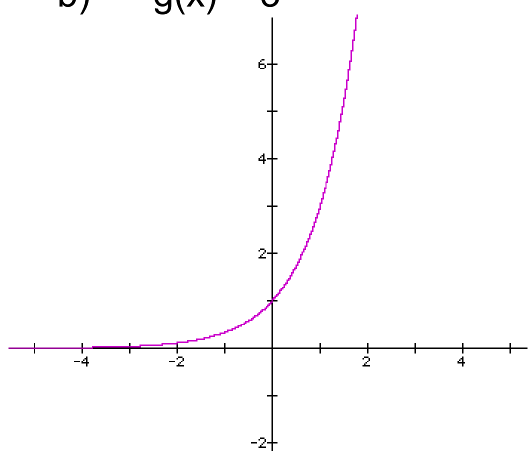
Solution:
We begin by evaluating the functions for various values of x. Use the x^y, y^x, or ^ key on your calculator to get these answers:

x	**– 3**	**– 2**	**– 1**	**0**	**1**	**2**	**3**
2^x	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8
3^x	$\frac{1}{27}$	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9	27
4^x	$\frac{1}{64}$	$\frac{1}{16}$	$\frac{1}{4}$	1	4	16	64

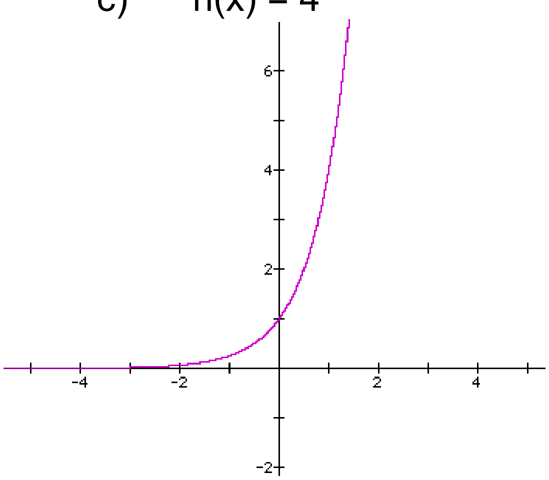
a) $f(x) = 2^x$



b) $g(x) = 3^x$



c) $h(x) = 4^x$



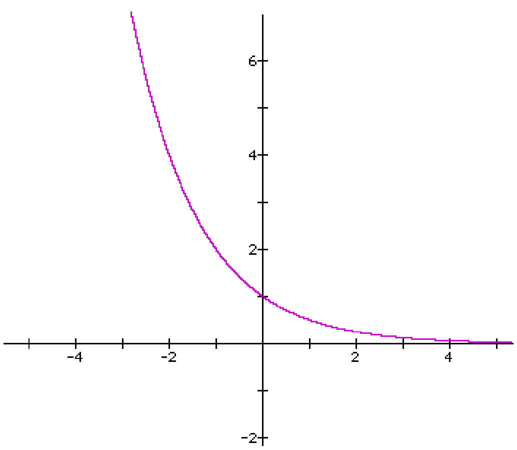
Notice that these functions have the same basic shape though as the base gets bigger, the graph gets "steeper." In general, if b > 1, b^x will have this shape.

Ex. 3 Make a table of values and graph the following:
a) $f(x) = 2^{-x}$ b) $g(x) = 3^{-x}$ c) $h(x) = 4^{-x}$

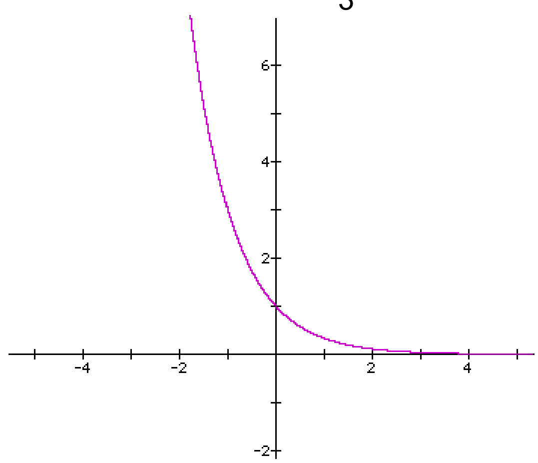
Solution:
We begin by evaluating the functions for various values of x. Use the x^y, y^x, or ^ key on your calculator to get these answers:

x	**– 3**	**– 2**	**– 1**	**0**	**1**	**2**	**3**
$2^{-x} = (\frac{1}{2})^x$	8	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$
$3^{-x} = (\frac{1}{3})^x$	27	9	3	1	$\frac{1}{3}$	$\frac{1}{9}$	$\frac{1}{27}$
$4^{-x} = (\frac{1}{4})^x$	64	16	4	1	$\frac{1}{4}$	$\frac{1}{16}$	$\frac{1}{64}$

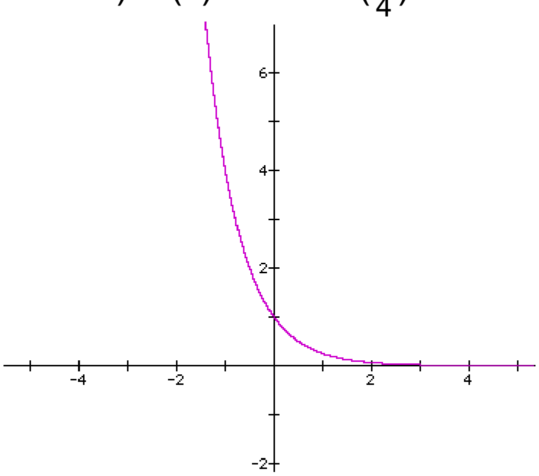
a) $f(x) = 2^{-x} = (\frac{1}{2})^x$



b) $g(x) = 3^{-x} = (\frac{1}{3})^x$



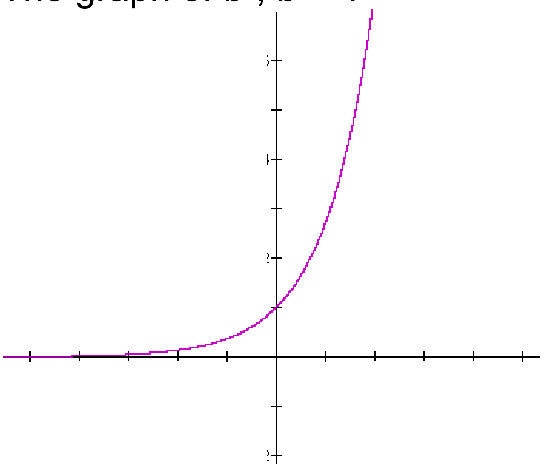
c) $h(x) = 4^{-x} = (\frac{1}{4})^x$



Notice that these functions have the same basic shape though as the base gets bigger, the graph gets "steeper." In general, if $0 < b < 1$, b^x will have this shape.

Notice that the graphs in example three are the graphs in example two reflected across the y-axis. Now, let's summarize what we know about exponential functions:

The graph of b^x, $b > 1$



Domain: $(-\infty, \infty)$

Range: $(0, \infty)$

Intercept: (0, 1)

Increasing

Concave Up

$y = 0$ is a horizontal Asymptote

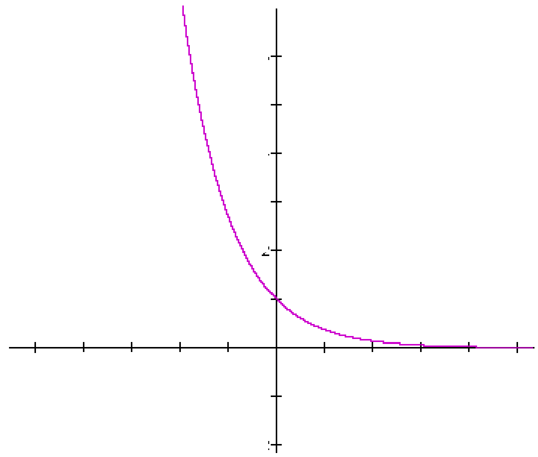
(as $x \to -\infty$, $b^x \to 0$)

Continuous

Exponential Growth

One - to - One

The graph of b^x, $0 < b < 1$



Domain: $(-\infty, \infty)$

Range: $(0, \infty)$

Intercept: (0, 1)

Decreasing

Concave Up

$y = 0$ is a horizontal Asymptote

(as $x \to \infty$, $b^x \to 0$)

Continuous

Exponential Decay

One - to - One

We can graph exponential functions using the techniques of shifting, stretching and reflecting. Let's consider the following examples

Ex. 4 Graph the following:

a) $f(x) = 2^{x+2}$

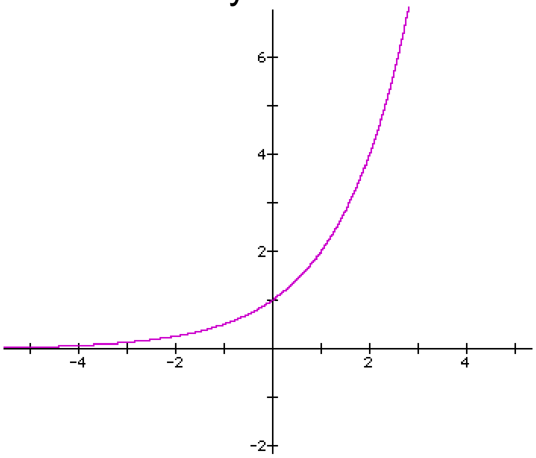
b) $g(x) = 2^x - 3$

c) $h(x) = 2^{x-1} + 2$

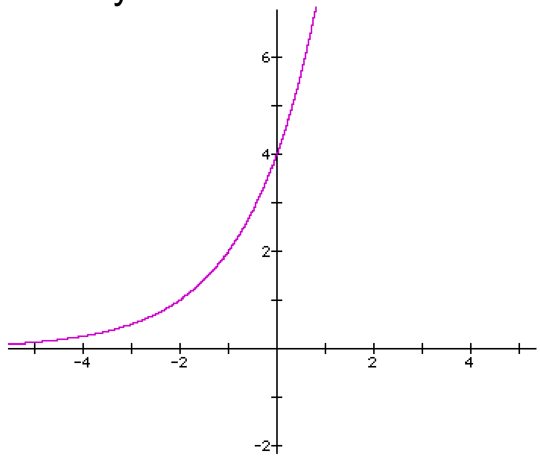
d) $f(x) = -2^x$

Solution:
a) $f(x) = 2^{x+2}$ is the graph of 2^x shifted 2 units to left:

Parent: $y = 2^x$

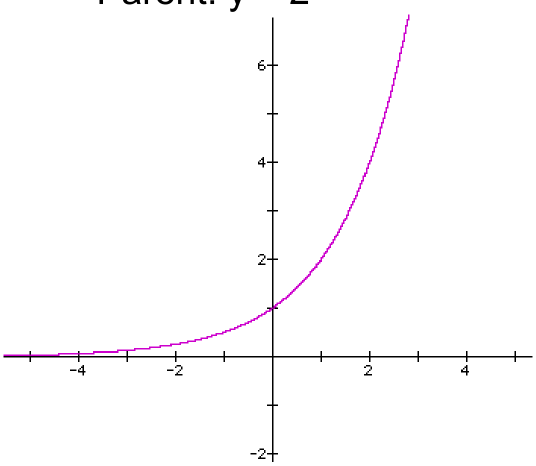


$y = 2^{x+2}$

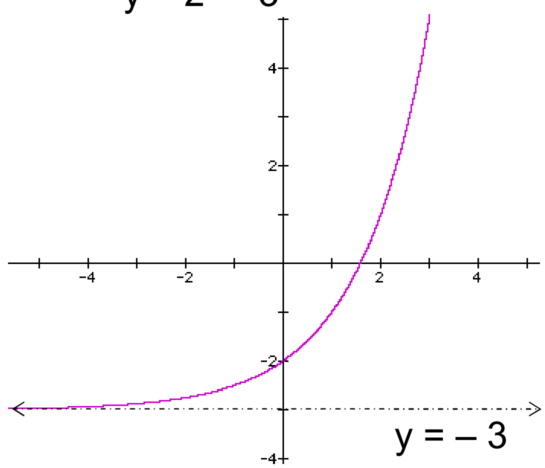


b) $g(x) = 2^x - 3$ is the graph of 2^x shifted down 3 units.

Parent: $y = 2^x$

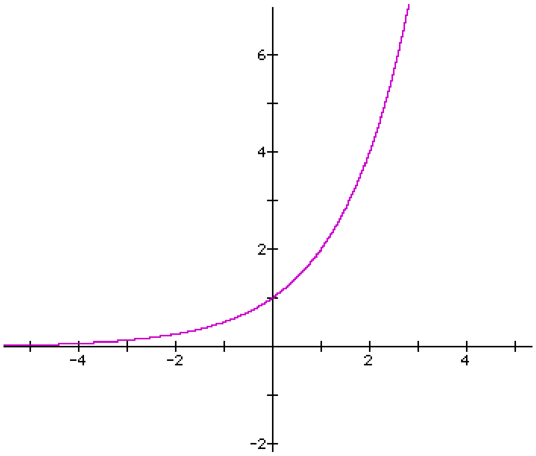


$y = 2^x - 3$

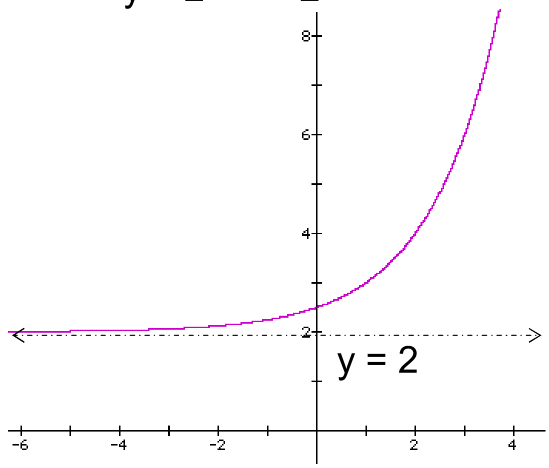


c) $h(x) = 2^{x-1} + 2$ is the graph of 2^x shifted up 2 units and to the right 1 unit.

Parent: $y = 2^x$

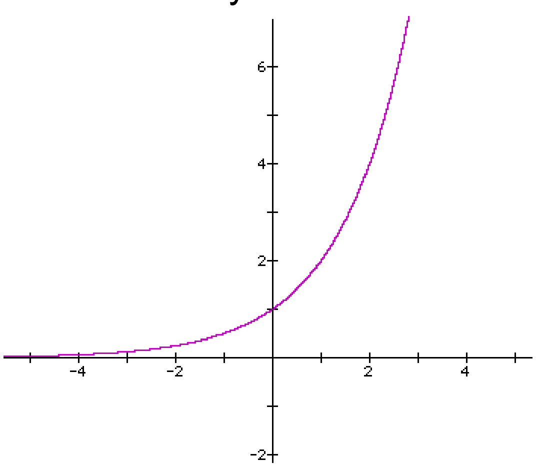


$y = 2^{x-1} + 2$

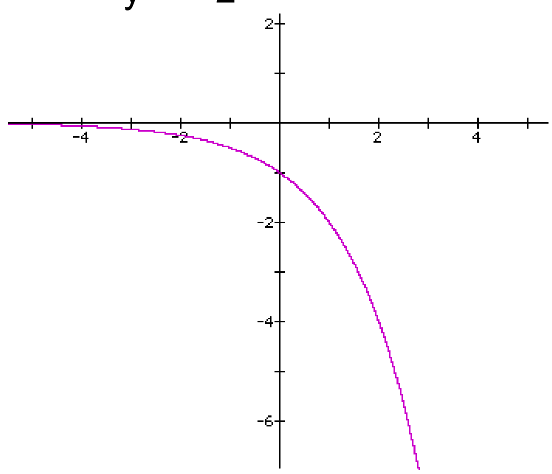


d) $f(x) = -2^x$ is the graph of 2^x reflected across the x-axis.

Parent: $y = 2^x$



$y = -2^x$



There is a special number that occurs with exponential functions. It is the natural base e. The number e is an irrational number that is approximately 2.71828182846.... It comes from examining the behavior of $[1 + \frac{1}{m}]^m$ as $m \to \infty$ (increases without bound). It may seem that $[1 + \frac{1}{m}]^m$ should approach 1, but if you plug in some large values for m, you will see this is not the case. For example, if m = 10,000, then

$[1 + \frac{1}{m}]^m = 2.718145936...$

It is important to be able to evaluate expressions involving e:

Ex. 5 Find

a) $e^{4.3}$ b) $e^{-1.2}$ c) $2e^3$

Solution:

On your calculator, you should have an e^x key. You might have to access it using the 2nd or shift key on your calculator. Most calculators have you type the exponent and then hit the e^x, but on some or the newer ones, you have to do the sequence backwards.

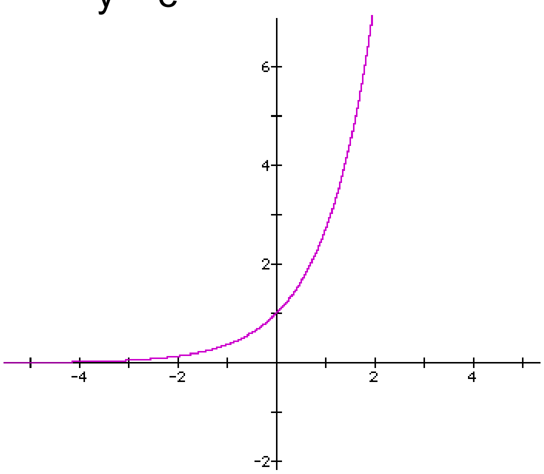
a) $e^{4.3} \approx 73.6997937$

b) $e^{-1.2} \approx 0.3011942119$

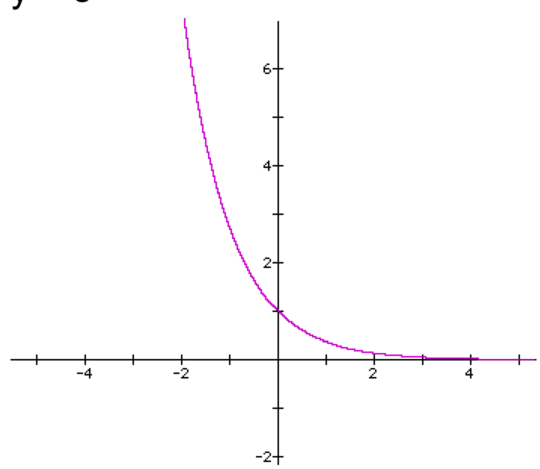
c) $2e^3 \approx 40.17107385$

Since e > 1, then the graph of e^x and e^{-x} will look like:

$y = e^x$



$y = e^{-x}$



An application of exponential functions is Compound Interest.

Formulas for compound interest:

After t years, the balance A (future value) in an account with principal P and annual percentage r (written as a decimal) compounded n times a year for t years is:

$$A = P\left(1+\frac{r}{n}\right)^{nt}$$

If you let n increase without bound so that interest is **compounded continuously**, then the formula is:

$$A = Pe^{rt}$$

Ex. 6 Suppose a total of \$15,000 is invested at an annual rate of 7% per year. Find the balance after 6 years if the interest is compounded:

a) quarterly b) monthly c) continuously

<u>Solution:</u>

a) If interest is compounded quarterly, then $n = 4$.
$P = \$15,000$, $r = 0.07$, and $t = 6$. Thus,
$$A = 15000\left(1+\frac{0.07}{4}\right)^{4(6)} = 15000(1.0175)^{24}$$
$$\approx 15000(1.516442786) \approx \$22,746.64.$$

b) If interest is compounded monthly, then $n = 12$.
$P = \$15,000$, $r = 0.07$, and $t = 6$. Thus,
$$A = 15000\left(1+\frac{0.07}{12}\right)^{12(6)} = 15000(1.00583333...)^{72}$$
$$\approx 15000(1.520105504) \approx \$22,801.58.$$

c) If interest is compounded continuously, then we plug in $P = \$15000$, $r = 0.07$, and $t = 6$ into $A = Pe^{rt}$:
$$A = 15000e^{0.07(6)} = 15000e^{0.42}$$
$$\approx 15000(1.521961556) \approx \$22,829.42.$$

Ex. 7 The population of a town increases according to the model $P(t) = 3300e^{0.0321t}$ where t is the time in years after 1990. Use the model to predict the population in the town in the year 2010.

<u>Solution:</u>

The year 2010 corresponds to t = 20 years (think: 2010 –1990). Thus, evaluating P at 20 yields:
$$P(20) = 3300e^{0.0321(20)} = 3300e^{0.642}$$
$$\approx 3300(1.900277637) \approx 6271 \text{ people.}$$

An application that involves exponential decay is the finding present value of an account with compound interest. You can think of the present value as how much money that you have in the account in the future is worth today. Since having \$30,000 in account in thirty years is not the same as having \$30,000 today. That amount is worth considerably less.

Present Value
Let B be the balance wanted in the future and r be the annual percentage (written as a decimal) compounded n times a year for t years. Then the present value P of the account is:

$$P = B\left(1+\frac{r}{n}\right)^{-nt}$$

If you let n increase without bound so that interest is **compounded continuously**, then the formula is:

$$P = Be^{-rt}$$

Ex. 8 Find the amount that Juanita should invest in an account paying 8% annual interest so that after 20 years, she will have \$50,000 if the interest is compounded

a) quarterly b) continuously

Solution:

a) If interest is compounded quarterly, then n = 4. B = \$50,000, r = 0.08, and t = 20. Thus,

$$P = 50000\left(1+\frac{0.08}{4}\right)^{-4(20)} = 50000(1.02)^{-80}$$

$$\approx 50000(0.2051097282) \approx \$10,255.49.$$

b) If interest is compounded continuously, then we plug in B = \$50000, r = 0.08, and t = 20 into $P = Be^{-rt}$:

$$P = 50000e^{-0.08(20)} = 50000e^{-1.6}$$

$$\approx 50000(0.201896518) \approx \$10,094.83.$$

Ex. 9 Find f(9) if $f(x) = e^{kx}$ and f(3) = 2.

Solution:

Since $f(3) = e^{k(3)} = e^{3k} = 2$, we can rewrite f(9) in terms of e^{3k} and then replace e^{3k} by 2:

$$f(9) = e^{9k} = (e^{3k})^3 = (2)^3 = 8.$$

Ex. 10 It is estimated that a population of a certain country grows exponential. If the population was 60 million people in 1991 and 90 million people in 1996, what will the population be in 2011?

Solution:

The function $Q(t) = Q_o e^{kt}$ is a model for exponential growth where Q_o is the initial population and k is the growth constant. In this example, Q_o will correspond to the population of 60 million in 1991 ($t = 0$). We can rewrite our function as:

$Q(t) = Q_o e^{kt} = 60e^{kt}$ where t is the number of years after 1991.

Since the population was 90 million in 1996 ($t = 5$), this means that:

$Q(5) = 60e^{k(5)} = 60e^{5k} = 90$

Solving for e^{5k} yields:

$60e^{5k} = 90$
$e^{5k} = 1.5$

Since 2011 corresponds to $t = 20$, we can find f(20):

$f(20) = 60e^{k(20)} = 60e^{20k}$.

But $e^{20k} = (e^{5k})^4 = (1.5)^4 = 5.0625$. Thus,

$f(20) = 60e^{20k} = 60(5.0625) = 303.75$ million people.

Thus, the country will have a population of 303.75 million people in 2011.

Ex. 11 The amount of a sample of radioactive substance remaining after t years is given by $Q(t) = Q_o e^{-0.005t}$. At the end of 300 years, 4,000 grams of the substance remained. How many grams were initially present?

Solution:

When t = 300 years, Q(t) = 4,000 grams. Plugging into $Q(t) = Q_o e^{-0.005t}$ yields:

$Q(300) = Q_o e^{-0.005(300)} = Q_o e^{-1.5} = 4000$

Now, we can solve for Q_o:

$Q_o e^{-1.5} = 4000$
$Q_o(0.223130160148) = 4000$
$Q_o \approx 17{,}926.76$ grams

Initially, there was about 17,926.76 grams of the substance present.

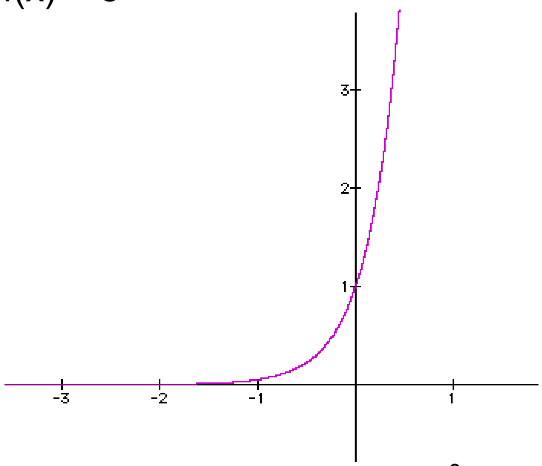
We will now look at some limits of exponential functions. It is important to keep in mind what the graph of a particular exponential function looks like.

Ex. 12 Evaluate the following limits if they exist:

a) $\lim_{x \to \infty} e^{3x}$ b) $\lim_{x \to -\infty} e^{3x}$

c) $\lim_{x \to \infty} e^{-3x}$ d) $\lim_{x \to -\infty} e^{-3x}$

Solution:
Recall that the graph of e^{3x} looks like:
$f(x) = e^{3x}$

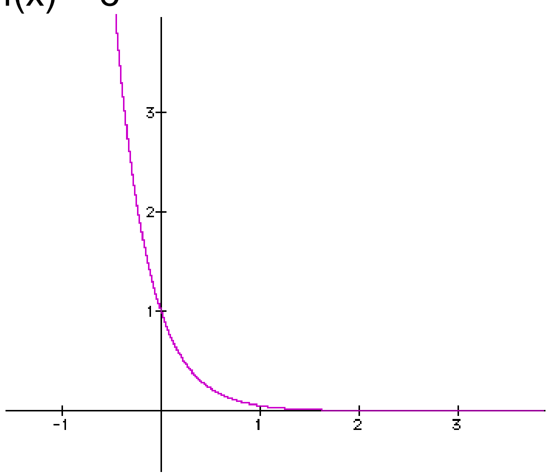


As x increases without bound, so does e^{3x}. But as x decreases without bound, e^{3x} goes to zero. Thus,

a) $\lim_{x \to \infty} e^{3x} = \infty$ and

b) $\lim_{x \to -\infty} e^{3x} = 0$.

Recall that the graph of e^{-3x} looks like:
$f(x) = e^{-3x}$



As x increases without bound, e^{-3x} goes to zero. But as x decreases without bound, e^{-3x} increases without bound. Thus,

c) $\lim_{x \to \infty} e^{-3x} = 0$ and

d) $\lim_{x \to -\infty} e^{-3x} = \infty$.

We will now examine how to differentiate exponential functions. The "proof" of how to differentiate an exponential function will be shown on the next section.

Derivative of the exponential function with base e

1) $\frac{d}{dx}[e^x] = e^x$ and

2) $\frac{d}{dx}[e^{f(x)}] = e^{f(x)} \bullet f'(x)$ Chain Rule

Differentiate the following:

Ex. 13 $f(x) = 3e^{4x+5}$

Solution:

$f'(x) = \frac{d}{dx}[3e^{4x+5}] = 3e^{4x+5} \bullet \frac{d}{dx}[4x+5] = 3e^{4x+5} \bullet [4] = 12e^{4x+5}$.

Ex. 14 $g(x) = e^{3x^2-6x+5}$

Solution:

$g'(x) = \frac{d}{dx}[e^{3x^2-6x+5}] = e^{3x^2-6x+5} \bullet \frac{d}{dx}[3x^2 - 6x + 5]$

$= e^{3x^2-6x+5}\ [6x - 6] = [6x - 6]e^{3x^2-6x+5}$.

Ex. 15 $h(x) = e^{\frac{1}{3x^2}}$

Solution:

$h'(x) = \frac{d}{dx}[e^{\frac{1}{3x^2}}] = e^{\frac{1}{3x^2}} \bullet \frac{d}{dx}[\frac{1}{3x^2}] = e^{\frac{1}{3x^2}} \bullet \frac{1}{3} \bullet \frac{d}{dx}[x^{-2}]$

$= e^{\frac{1}{3x^2}} \bullet \frac{1}{3} \bullet [-2x^{-3}] = e^{\frac{1}{3x^2}} \bullet \frac{1}{3} \bullet \frac{-2}{x^3} = \frac{-2e^{\frac{1}{3x^2}}}{3x^3}$

Ex. 16 $Q(t) = Q_o e^{kt}$

Solution:

$\frac{d}{dt}[Q(t)] = \frac{d}{dt}[Q_o e^{kt}]$

$Q'(t) = Q_o e^{kt} \bullet \frac{d}{dt}[kt] = Q_o e^{kt} \bullet k = k \bullet Q_o e^{kt}$.

Observe that in the last example, since $Q(t) = Q_o e^{kt}$, we can rewrite Q '(t) as : $Q'(t) = k \bullet Q_o e^{kt} = k \bullet Q(t)$. This says that the growth rate of an exponential function is proportional to its size. Similarly, since $\%Rate\Delta Q = 100\% \bullet \frac{Q'(t)}{Q(t)}$, we can substitute $Q'(t) = k \bullet Q(t)$ to get : $\%Rate\Delta Q = 100\% \bullet \frac{k \bullet Q(t)}{Q(t)} = 100\% \bullet k$. We can summarize these results below:

Exponential growth

If Q(t) grows exponentially according to the model $Q(t) = Q_o e^{kt}$, then:

1) $Q'(t) = k \bullet Q(t)$
2) $\%Rate\Delta Q = 100\% \bullet k$

Ex. 17 Suppose money is deposited in an account paying 9% interest compounded continuously. Find the percentage rate of change of the balance with respect to time.

Solution:

Since $\%Rate\Delta Q = 100\% \bullet k$, then $\%Rate\Delta Q = 9\%$.