Section 4.2 – Logarithmic Functions & Applications

Recall that exponential functions are one-to-one since every horizontal line passes through at most one point on the graph of $y = b^x$. So, an exponential function has an inverse. We call this inverse function a logarithmic function.

Definition of a Logarithmic Function
For $x > 0$ and $b > 0$ and $b \neq 1$, then

$y = f(x) = \log_b(x)$ if and only if $x = b^y$.

This called the logarithmic function with base b.
Note, the domain of $\log_b(x)$ is $(0, \infty)$.

Think of the logarithmic functions as asking the question:
"By what power f(x) do we have to raise the base b to get x?"

Ex. 1 Find

a) $\log_{10}(100)$ b) $\log_2(16)$ c) $\log_3(\frac{1}{81})$
d) $\log_5(5)$ e) $\log_7(1)$ f) $\log_2(0)$

Solution:

a) For $\log_{10}(100)$, ask the question: "By what power f(x) do we have to raise the base 10 to get 100?" The answer is 2. Thus, since $10^2 = 100$, then $\log_{10}(100) = 2$.

b) For $\log_2(16)$, ask the question: "By what power f(x) do we have to raise the base 2 to get 16?" The answer is 4. Thus, since $2^4 = 16$, then $\log_2(16) = 4$.

c) For $\log_3(\frac{1}{81})$, ask the question: "By what power f(x) do we have to raise the base 3 to get $\frac{1}{81}$?" Since $\frac{1}{81} = 81^{-1} = (3^4)^{-1} = 3^{-4}$, the answer is – 4. Thus, since $3^{-4} = \frac{1}{81}$, then $\log_3(\frac{1}{81}) = -4$.

d) For $\log_5(5)$, ask the question: "By what power f(x) do we have to raise the base 5 to get 5?" The answer is 1. Thus, since $5^1 = 5$, then $\log_5(5) = 1$.

e) For $\log_7(1)$, ask the question: "By what power f(x) do we have to raise the base 7 to get 1?" The answer is 0. Thus, since $7^0 = 1$, then $\log_7(1) = 0$.

f) The domain of the logarithmic function is $(0, \infty)$. Thus, $\log_2(0)$ is undefined.

Ex. 2 Solve the following:

a) $\log_4(x) = 2$ b) $\log_{25}(x) = \frac{1}{2}$ c) $\log_{7.5}(x) = -0.2$

Solution:

a) Let's rewrite the function as an exponential function since $y = f(x) = \log_b(x)$ if and only if $x = b^y$:

$\log_4(x) = 2$ if and only if $x = 4^2 = 16$.

Thus, x = 16.

b) Let's rewrite the function as an exponential function since $y = f(x) = \log_b(x)$ if and only if $x = b^y$:

$\log_{25}(x) = \frac{1}{2}$ if and only if $x = 25^{1/2} = \sqrt{25} = 5$.

Thus, x = 5.

c) Let's rewrite the function as an exponential function since $y = f(x) = \log_b(x)$ if and only if $x = b^y$:

$\log_{7.5}(x) = -0.2$ if and only if $x = 7.5^{-0.2} \approx 0.6683$.

Thus, $x \approx 0.6683$.

If have two special bases of logarithms that we usually work with. One is base 10 and the other is base e. Both of these functions have keys on a scientific calculator.

If b = 10, then $\log_{10}(x)$ is called the common log. It is usually denoted as log (x). Thus, if there is no base written, assume that it is the common log.

If b = e, then $\log_e(x)$ is called the natural log. It is usually denoted as ln (x).

Ex. 3 Evaluate the following:

a) $\ln(e^2)$ b) ln (e) c) ln (1) d) ln (7) e) ln (0.024)

Solution:
a) Ask: "By what power do we have to raise e to get e^2?" The answer is 2. Thus, $\ln(e^2) = 2$.

b) Ask: "By what power do we have to raise e to get e?" The answer is 1. Thus, $\ln(e) = 1$.

c) Ask: "By what power do we have to raise e to get 1?" The answer is 0. Thus, $\ln(1) = 0$.

d) This one we cannot do in our head. Here, we will need to use a calculator. Some calculators, you type in the number and then hit the ln button. Others, you have to reverse the order.

$\ln(7) \approx 1.945910149$

e) Again, we have to use our calculator:

$\ln(0.024) \approx -3.729701449$

Properties of Logarithms:

1) $\log_b(1) = 0$ since $b^0 = 1$.
2) $\log_b(b) = 1$ since $b^1 = b$.
3) $\log_b(b^x) = x$ since $b^x = b^x$.
4) If $\log_b(x) = \log_b(y)$, then $x = y$.

Since exponential and logarithmic functions are inverse functions, the graph of a log function should be the graph of the respective exponential function reflected across the line $y = x$. To see this, let's work the following example:

Ex. 4 Make a table of values and sketch the graph of:

a) $f(x) = \log_2(x)$ b) $g(x) = \log_4(x)$

c) $h(x) = \log_{1/2}(x)$ d) $k(x) = \log_{1/4}(x)$

Solution:

The graph of $\log_2(x)$ should be the graph of $y = 2^x$ reflected across the line $y = x$; the graph of $\log_4(x)$ should be the graph of $y = 4^x$ reflected across the line $y = x$; and so on. Let's make some tables:

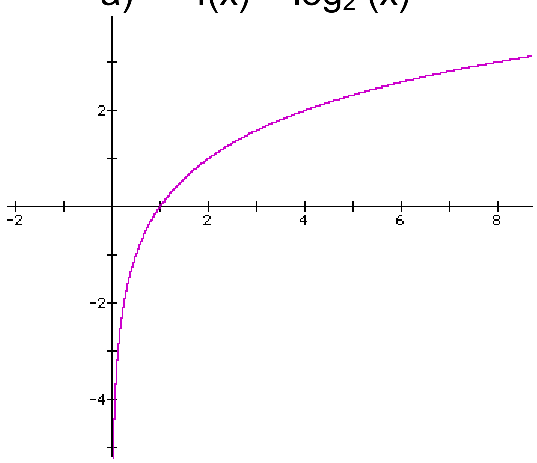
x	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8
$\log_2 (x)$	-3	-2	-1	0	1	2	3

x	$\frac{1}{64}$	$\frac{1}{16}$	$\frac{1}{4}$	1	4	16	64
$\log_4 (x)$	-3	-2	-1	0	1	2	3

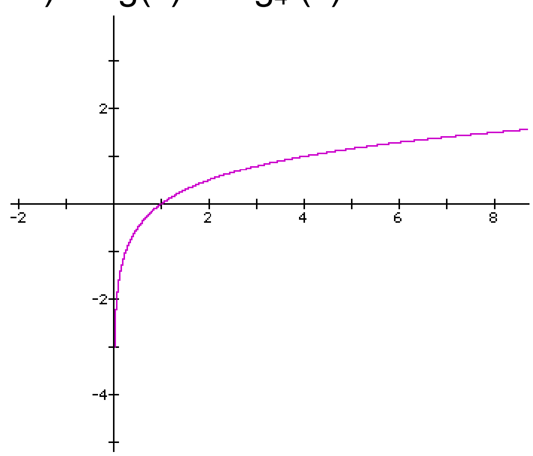
x	8	4	2	1	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$
$\log_{1/2} (x)$	-3	-2	-1	0	1	2	3

x	64	16	4	1	$\frac{1}{64}$	$\frac{1}{16}$	$\frac{1}{4}$
$\log_{1/4} (x)$	-3	-2	-1	0	1	2	3

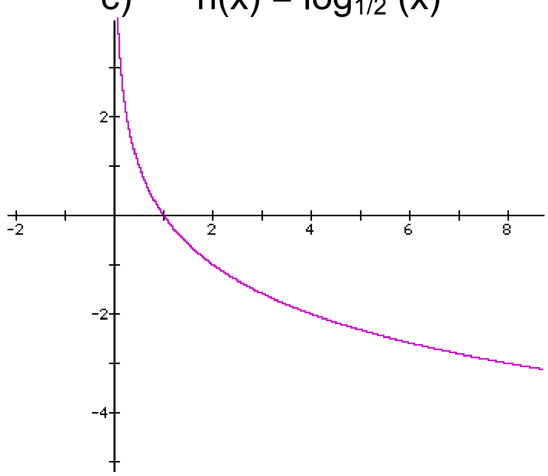
a) $f(x) = \log_2 (x)$



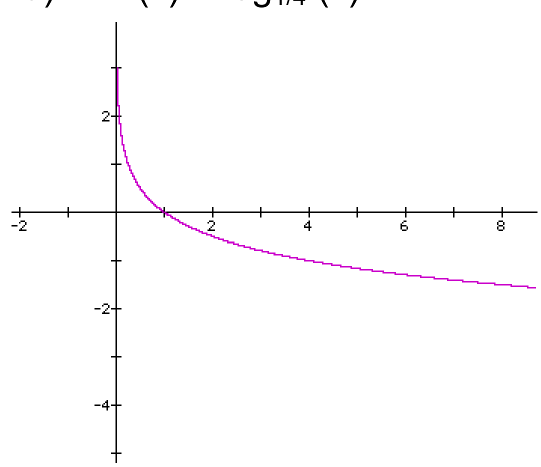
b) $g(x) = \log_4 (x)$



c) $h(x) = \log_{1/2} (x)$

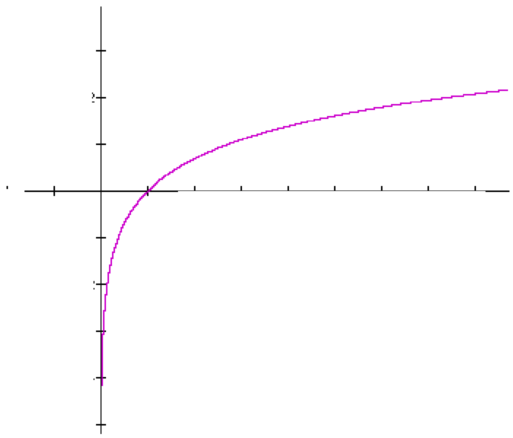


d) $k(x) = \log_{1/4} (x)$

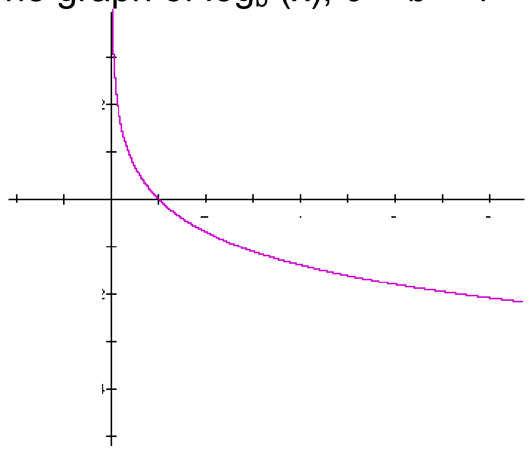


Notice that the graphs in example 4 c & d are the graphs in example 4 a & d reflected across the x-axis. Now, let's summarize what we know about logarithmic functions:

The graph of $\log_b(x)$, $b > 1$



The graph of $\log_b(x)$, $0 < b < 1$



The graph of $\log_b(x)$, $b > 1$	The graph of $\log_b(x)$, $0 < b < 1$
Domain: $(0, \infty)$	Domain: $(0, \infty)$
Range: $(-\infty, \infty)$	Range: $(-\infty, \infty)$
Intercept: (1, 0)	Intercept: (1, 0)
Increasing	Decreasing
Concave Down	Concave Up
$x = 0$ is a Vertical Asymptote	$x = 0$ is a Vertical Asymptote
(as $x \to 0^+$, $\log_b(x) \to -\infty$)	(as $x \to 0^+$, $\log_b(x) \to \infty$)
Continuous	Continuous
One - to - One	One - to - One

We can also graph log functions using the techniques of shifting stretching and reflecting. Consider the following example:

Ex. 5 Sketch the graph of:

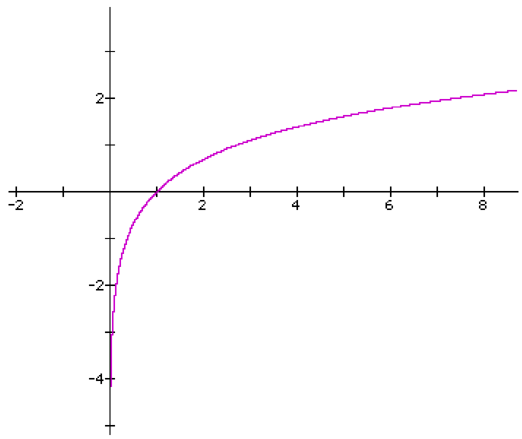
a) $g(x) = \ln(x - 2) + 3$ b) $h(x) = -2 \cdot \log_{1/2}(x + 3)$

<u>Solution:</u>

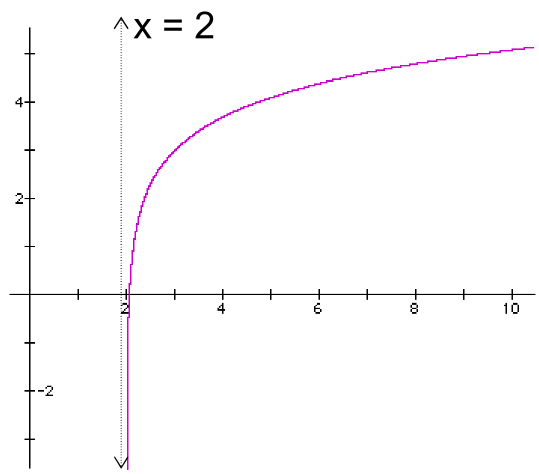
a) Since $e > 1$, the parent function will look like the graph of the function above on the left. The graph

$g(x) = \ln(x - 2) + 3$ is the graph of $\ln(x)$ shifted up 3 units and to the right two units.

Parent: $y = \ln(x)$



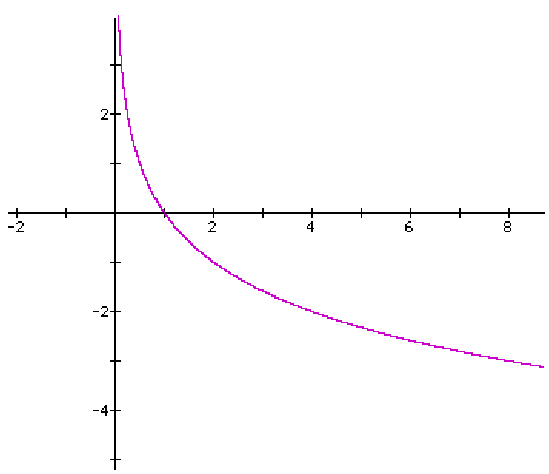
$g(x) = \ln(x - 2) + 3$



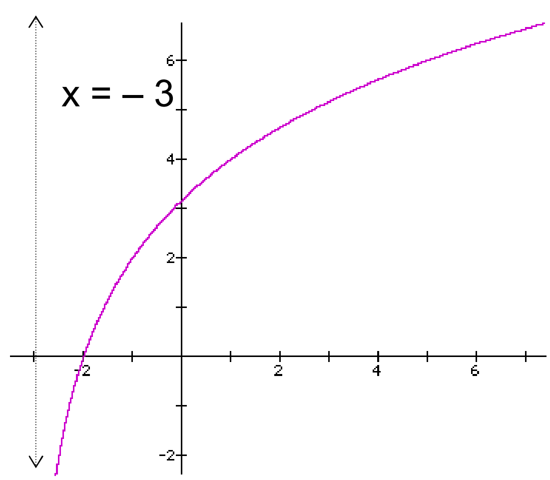
Note that the domain of g is $x > 2$ a since $x - 2 > 0$.

b) The graph of $h(x) = -2 \cdot \log_{1/2}(x + 3)$ is the graph of $\log_{1/2}(x)$ reflected across the x-axis, stretched by a factor of two and shifted to the left three units.

Parent: $y = \log_{1/2}(x)$



$h(x) = -2 \cdot \log_{1/2}(x + 3)$



Note that the domain of h is $x > -3$ a since $x + 3 > 0$.

Most calculators have only two types of keys for evaluating log functions, the common log and the natural log. To evaluate logs of other bases, we use the change of base formula:

Change of Base Formula

Let a, b, and x be positive real numbers such that a ≠ 0 and b ≠ 0, then we can find $\log_a(x)$ by:

$$\log_a(x) = \frac{\log_b(x)}{\log_b(a)}$$

Ex. 6 Find to the nearest thousandth:

a) $\log_2(0.35)$ b) $\log_5(10.3)$ $\log_{1/3}(0.2)$

Solution:

a) We need to use the change of base formula with either the common log or the natural log:

$$\log_2(0.35) = \frac{\log(0.35)}{\log(2)} \approx \frac{-0.4559319556}{0.3010299957} \approx -1.515$$

If we use the natural log, we get:

$$\log_2(0.35) = \frac{\ln(0.35)}{\ln(2)} \approx \frac{-1.049822124}{0.6931471806} \approx -1.515$$

Either way, we get the same thing.

b) If we use the natural log, we get:

$$\log_5(10.3) = \frac{\ln(10.3)}{\ln(5)} \approx \frac{2.332143895}{1.609437912} \approx 1.449$$

c) If we use the natural log, we get:

$$\log_{1/3}(0.2) = \frac{\ln(0.2)}{\ln(1/3)} \approx \frac{-1.609437912}{-1.098612289} \approx 1.465$$

More Properties of Logarithms

Let b > 0 and b ≠ 1 and let n be a real number. If u and v are positive numbers, then the following properties are true:

1) $\log_b(u \bullet v) = \log_b(u) + \log_b(v)$

2) $\log_b\left(\frac{u}{v}\right) = \log_b(u) - \log_b(v)$

3) $\log_b(u^n) = n \bullet \log_b(u)$

Caution: $\log_b(u + v) \neq \log_b(u) + \log_b(v)$ and
$\log_b(u - v) \neq \log_b(u) - \log_b(v)$

If you let $u = b^x$ and $v = b^y$, then property #1 becomes:
$\log_b(u \bullet v) = \log_b(b^x \bullet b^y) = \log_b(b^{x+y}) = x + y = \log_b(b^x) + \log_b(b^y)$
$= \log_b(u) + \log_b(v)$. A similar proof works for Prop. #2 and #3.

Ex. 7 Given $\log_b(6) \approx 1.2$ and $\log_b(7) \approx 1.7$, find:

a) $\log_b(42)$ b) $\log_b\left(\sqrt{\frac{6}{7}}\right)$ c) $\log_b(6b^2)$

Solution:

We need to our properties of logs to rewrite each problem in terms of $\log_b(6)$ and $\log_b(7)$.

a) $\log_b(42) = \log_b(6\text{•}7) = \log_b(6) + \log_b(7)$ Prop. 1

$\approx 1.2 + 1.7 = 2.9.$

b) $\log_b\left(\sqrt{\frac{6}{7}}\right) = \log_b\left(\left[\frac{6}{7}\right]^{1/2}\right) = \frac{1}{2}\text{•}\log_b\left(\frac{6}{7}\right)$ Prop. 3

$= \frac{1}{2}[\log_b(6) - \log_b(7)]$ Prop. 2

$\approx \frac{1}{2}[1.2 - 1.7] = -0.25.$

c) $\log_b(6b^2) = \log_b(6) + \log_b(b^2)$ Prop. 1

$= \log_b(6) + 2\text{•}\log_b(b)$ Prop. 3

$= 1.2 + 2\text{•}1$

$= 3.2.$

Ex. 8 Write as the sum and/or difference of logs.

a) $\ln\left(\frac{x^2y^3}{z}\right)$ b) $\ln\left(\frac{5x}{\sqrt{x^2+1}}\right)$

Solution:

Again, we need to use our properties of logs:

a) $\ln\left(\frac{x^2y^3}{z}\right) = \ln(x^2y^3) - \ln(z)$ Prop. 2

$= \ln(x^2) + \ln(y^3) - \ln(z)$ Prop. 1

$= 2\text{•}\ln(x) + 3\text{•}\ln(y) - \ln(z)$ Prop. 3

b) $\ln\left(\frac{5x}{\sqrt{x^2+1}}\right) = \ln(5x) - \ln[(x^2+1)^{1/2}]$ Prop. 2

$= \ln(5x) - \frac{1}{2}\text{•}\ln(x^2+1)$ Prop. 3

$= \ln(5) + \ln(x) - \frac{1}{2}\text{•}\ln(x^2+1)$ Prop. 1

Ex. 9 Write as the logarithm of a single expression:

a) $\frac{2}{3}\text{•}\ln(x) + \ln(y) - 3\ln(z)$

b) $5\text{•}\ln(x+1) - 3\text{•}\ln(x) + \ln(x+2)$

Solution:

a) $\frac{2}{3}\bullet\ln(x) + \ln(y) - 3\ln(z)$
$= \ln(x^{2/3}) + \ln(y) - \ln(z^3)$ Prop. 3
$= \ln(x^{2/3}y) - \ln(z^3)$ Prop. 1
$= \ln\left(\frac{x^{2/3}y}{z^3}\right)$ Prop. 2
$= \ln\left(\frac{\sqrt[3]{x^2}\,y}{z^3}\right)$

b) $5\bullet\ln(x+1) - 3\bullet\ln(x) + \ln(x+2)$
$= \ln[(x+1)^5] - \ln(x^3) + \ln(x+2)$ Prop. 3
$= \ln[(x+1)^5(x+2)] - \ln(x^3)$ Prop. 1
$= \ln\left[\frac{(x+1)^5(x+2)}{x^3}\right]$ Prop. 2

Ex. 10 Simplify:

a) $\ln(\sqrt{e})$ b) $e^{2\ln(3)}$ c) $\ln\left(\frac{e^3\sqrt{e}}{e^{1/3}}\right)$

Solution:

For all these problems, we will use the fact that e^x and $\ln(x)$ are inverse functions. In other words, $e^{\ln(\#)} = \#$ and $\ln(e^{\#}) = \#$.

a) $\ln(\sqrt{e}) = \ln(e^{1/2}) = \frac{1}{2}$ (Inverse functions)

b) $e^{2\ln(3)} = e^{\ln(3^2)}$ Prop. 3
$= e^{\ln(9)} = 9$ (Inverse functions)

c) $\ln\left(\frac{e^3\sqrt{e}}{e^{1/3}}\right) = \ln(e^3\sqrt{e}) - \ln(e^{1/3})$ Prop. 2
$= \ln(e^3) + \ln(\sqrt{e}) - \ln(e^{1/3})$ Prop. 1
$= 3 + 1/2 - 1/3 = 3\frac{1}{6}$.

Solve the following for x:

Ex. 11 $\ln(x) = 2[\ln(3) - \ln(5)]$

Solution:
Here, we will use our properties of logs:
$\ln(x) = 2[\ln(3) - \ln(5)]$
$\ln(x) = 2[\ln(3/5)]$ Prop. 2
$\ln(x) = \ln[(3/5)^2]$ Prop. 3
$\ln(x) = \ln[9/25]$
Since ln (x) is one-to-one, then
$x = \frac{9}{25}$

Ex. 12 a) $e^{10x} = 6$ b) $5 \cdot \ln(x) = 10$
Solution:
a) Since ln(x) is one-to-one, we can take the ln of both sides of the equation:

$e^{10x} = 6$
$\mathbf{ln}(e^{10x}) = \mathbf{ln}(6)$
$10x = \ln(6)$
$x = \frac{\ln(6)}{10} \approx 0.1792.$

b) We begin by solving for ln (x). Then, since e^x is one-to-one, we can raise e by the quantities on both sides of the equation:

$5 \cdot \ln(x) = 10$
$\ln(x) = 2$
$e^{\ln(x)} = e^2$
$x = e^2 \approx 7.3891.$

Ex. 13 Suppose a certain amount is invested at 7% compounded continuously. How long would it take for the money to double?
Solution:
Let Q_o be the initial amount invested. Then, double that amount would be $2Q_o$. Thus, our equation becomes:

$2Q_o = Q_o e^{kt}$ (divide by Q_o)
$2 = e^{kt}$

Take the ln of both sides:

$\mathbf{ln}(2) = \mathbf{ln}(e^{kt})$
$\ln(2) = kt$ Now, divide by k

Thus, for doubling time, we can use the formula:

(Rule of 70) $t = \frac{\ln(2)}{k}$ or $k = \frac{\ln(2)}{t}$

Since the interest rate is 7%, then k = 0.07. Hence,

$$t = \frac{\ln(2)}{0.07} \approx 9.902.$$

It will take about 9.902 years for the money to double.

Ex. 14 A publisher estimates that if x thousand complimentary copies of a certain textbook are distributed to instructors, the first-year sales of the textbook will be approximately $f(x) = 20 - 15e^{-0.2x}$ thousand copies. How many complementary copies should the publisher distribute to generate the first-year sales of 12,000 copies?

Solution:

Since the target sales is 12,000 copies, we will set f(x) = 12 and solve for x:

$$20 - 15e^{-0.2x} = 12$$
$$-15e^{-0.2x} = -8$$
$$e^{-0.2x} = \frac{8}{15}$$

Now, take the ln of both sides:

$$\mathbf{ln}\ (e^{-0.2x}) = \mathbf{ln}\ (\tfrac{8}{15})$$
$$-0.2x = \ln(\tfrac{8}{15})$$
$$x = \frac{\ln\left(\frac{8}{15}\right)}{-0.2} \approx 3.14304$$

Thus, about 3,143 complementary copies will need to be distributed.

Ex. 15 The total sales of a company was \$100 million in 1995 and \$150 million in 1998. If the sales are growing exponentially, find the rate of growth per year and predict what the total sales will be in 2005.

Solution:

Let $Q_o = 100$. Then, when t = 3, Q(3) = 150. We can use this equation to solve for k:

$$Q(3) = 100e^{k(3)} = 100e^{3k} = 150$$
$$e^{3k} = 1.5$$

Take the ln of both sides:

$$\ln(e^{3k}) = \ln(1.5)$$

$3k = \ln(1.5)$

$k = \frac{\ln(1.5)}{3} \approx 0.13516$

So, the sales are growing at a rate of about 13.516% per year. Thus, our function can now be written as:

$Q(t) = 100e^{0.13516t}$

Since 2005 corresponds to t = 10, then

$Q(10) = 100e^{0.13516(10)} = 100e^{1.3516} \approx 372.7$

Hence, in 2005, the sales should be about $372.7 million.

Ex. 16 The half-life for carbon-14 (^{14}C) is 5,730 years. If a scientist found a fossil in which the ratio of ^{14}C to ^{12}C is $\frac{1}{3}$ of the ratio found in the atmosphere. How old is the fossil?

<u>Solution:</u>

For half-life, we can use the same formula as for doubling time. Since the half-life is 5,730 years, the decay constant is:

$k = \frac{\ln(2)}{t} = \frac{\ln(2)}{5730} \approx 1.20986 \times 10^{-4}$

Since the ratio is $\frac{1}{3}$ of the ratio in the atmosphere, the amount of ^{14}C in the fossil is $\frac{1}{3}$ of the original amount.

Thus,

$Q(t) = Q_o e^{-kt}$

$\frac{1}{3}Q_o = Q_o e^{-0.000120968t}$ (divide by Q_o)

$\frac{1}{3} = e^{-0.000120968t}$ (take the ln of both sides)

$\ln(\frac{1}{3}) = \ln(e^{-0.000120968t})$

$\ln(\frac{1}{3}) = -0.000120968t$

$t = \frac{\ln(\frac{1}{3})}{-0.000120968} \approx 9082.$

Hence, the fossil is about 9,082 years old.

Carbon 14 dating is only used for objects that are less than 50,000 years old. Beyond that, there is too little Carbon 14 to detect.

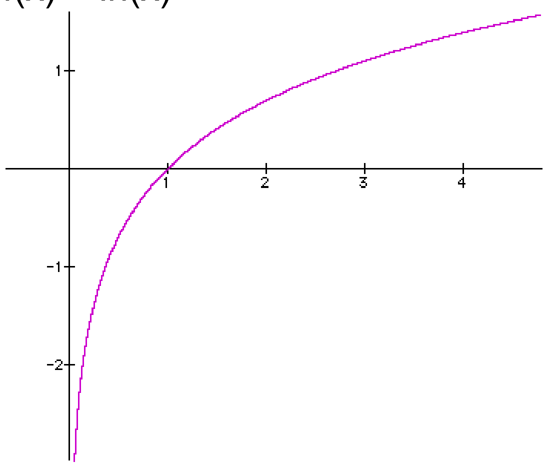
Ex. 17 Evaluate the following limits if they exist:

a) $\lim\limits_{x \to \infty} \ln(x)$ b) $\lim\limits_{x \to 0^+} \ln(x)$

<u>Solution:</u>

Recall that the graph of ln(x) looks like:

f(x) = ln(x)



As x increases without bound, so does ln(x). But as x goes to zero from the positive side, ln(x) decreases without bound. Thus,

a) $\lim\limits_{x \to \infty} \ln(x) = \infty$ &

b) $\lim\limits_{x \to 0^+} \ln(x) = -\infty$.

We will now examine how to differentiate exponential and logarithmic functions. We will begin with differentiate ln (x). Since the natural log is not a polynomial function, we will need to use the definition of the derivative and the fact that $\lim\limits_{m \to \infty} (1 + \frac{1}{m})^m = e$.

Ex. 18 Find the derivative of ln (x) using the definition of the derivative.

<u>Solution:</u>

If y = ln (x), then

$$y' = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\ln(x+h) - \ln(x)}{h}$$

$$= \lim_{h \to 0} \frac{1}{h} \bullet \{\ln(x + h) - \ln(x)\}$$

$$= \lim_{h \to 0} \frac{1}{h} \bullet \{\ln(\frac{x+h}{x})\} \qquad \text{Prop. \#2 of Logs.}$$

$$= \lim_{h \to 0} \ln\{\left(\frac{x+h}{x}\right)^{\frac{1}{h}}\} \qquad \text{Prop. \#3 of Logs.}$$

$$= \lim_{h \to 0} \ln\{\left(1 + \frac{h}{x}\right)^{\frac{1}{h}}\}$$

Now, we can make a substitution. Let $m = \frac{x}{h}$, then $\frac{m}{x} = \frac{1}{h}$ and $\frac{1}{m} = \frac{h}{x}$. Also, $h \to 0$ implies $m \to \infty$.

Thus,

$$\lim_{h \to 0} \ln \left\{\left(1+\frac{h}{x}\right)^{\frac{1}{h}}\right\} = \lim_{m\to\infty} \ln \left\{\left(1+\frac{1}{m}\right)^{\frac{m}{x}}\right\}$$

$$= \lim_{m\to\infty} \ln \left\{\left[\left(1+\frac{1}{m}\right)^{m}\right]^{\frac{1}{x}}\right\}.$$

But, $\lim\limits_{m\to\infty} (1 + \frac{1}{m})^m = e$, so

$$\lim_{m\to\infty} \ln \left[\left[\left(1+\frac{1}{m}\right)^{m}\right]^{\frac{1}{x}}\right] = \ln \{[e]^{\frac{1}{x}}\} = \frac{1}{x} \bullet \ln \{e\} \quad \text{Prop. \#3}$$

$$= \frac{1}{x}\bullet 1 = \frac{1}{x}.$$

Derivative of the natural logarithm
If $x > 0$ and $f(x) > 0$, then

1) $\frac{d}{dx}[\ln (x)] = \frac{1}{x}$ and
2) $\frac{d}{dx}[\ln (f(x))] = \frac{1}{f(x)} \bullet \frac{d}{dx}[f(x)] = \frac{f'(x)}{f(x)}$ (chain rule)

<u>Differentiate the following functions:</u>

Ex. 19 $y = \ln (2x)$

<u>Solution:</u>

$$y' = \frac{d}{dx}[\ln (2x)] = \frac{1}{2x} \bullet \frac{(x^2\, \check{S}3)^3}{\sqrt[6]{3x+5}} [2x] = \frac{1}{2x}\bullet 2 = \frac{1}{x}$$

Ex. 20 $y = \ln (x^3 + 3)$

<u>Solution:</u>

$$y' = \frac{d}{dx}[\ln (x^3 + 3)] = \frac{1}{x^3+3} \bullet \frac{d}{dx}[x^3 + 3] = \frac{1}{x^3+3}\bullet 3x^2 = \frac{3x^2}{x^3+3}$$

Ex. 21 $y = \ln [(x + 3)^3\bullet(x - 5)^4]$

<u>Solution:</u>

First use the properties of logs to simplify y:

$y = \ln [(x + 3)^3\bullet(x - 5)^4]$

$= \ln [(x + 3)^3] + \ln [(x - 5)^4]$ Prop. 1

$= 3 \cdot \ln(x + 3) + 4 \cdot \ln(x - 5)$ Prop. 3

Now differentiate:

$y' = 3 \cdot \frac{d}{dx}[\ln(x + 3)] + 4 \cdot \frac{d}{dx}[\ln(x - 5)]$

$= 3 \cdot \frac{1}{x+3} \cdot [1] + 4 \cdot \frac{1}{x-5} \cdot [1] = \frac{3}{x+3} + \frac{4}{x-5}$.

Ex. 22 $y = \ln\left(\frac{3x^2+1}{\sqrt{2x-1}}\right)$

Solution:

First use the properties of logs to simplify y:

$y = \ln\left(\frac{3x^2+1}{\sqrt{2x-1}}\right) = \ln(3x^2 + 1) - \ln(\sqrt{2x-1})$ Prop. #2

$= \ln(3x^2 + 1) - \ln([2x - 1]^{1/2})$

$= \ln(3x^2 + 1) - \frac{1}{2}\ln(2x - 1)$ Prop. #3

Now, we can differentiate:

$y' = \frac{d}{dx}[\ln(3x^2 + 1)] - \frac{1}{2} \cdot \frac{d}{dx}[\ln(2x - 1)]$

$= \frac{1}{3x^2+1} \cdot \frac{d}{dx}[3x^2 + 1] - \frac{1}{2} \cdot \frac{1}{2x-1} \cdot \frac{d}{dx}[2x - 1]$

$= \frac{1}{3x^2+1} \cdot [6x] - \frac{1}{2} \cdot \frac{1}{2x-1} \cdot [2]$

$= \frac{6x}{3x^2+1} - \frac{1}{2x-1}$.

Logarithmic Differentiation:

Sometimes it is easier to take the natural log of both sides of an equation before differentiating. Let's look at some examples:

Ex. 23 $f(x) = \frac{(x^2-3)^3}{\sqrt[6]{3x+5}}$

Solution:

We begin by taking the natural log of both sides and simplify the right side of the equation:

$\mathbf{ln}\,[f(x)] = \mathbf{ln}\,\left[\frac{(x^2-3)^3}{\sqrt[6]{3x+5}}\right]$

$\ln[f(x)] = \ln[(x^2 - 3)^3] - \ln[\sqrt[6]{3x+5}]$ Prop. #2

$\ln[f(x)] = \ln[(x^2 - 3)^3] - \ln[(3x + 5)^{1/6}]$

$\ln [f(x)] = 3 \bullet \ln (x^2 - 3) - \frac{1}{6} \bullet \ln (3x + 5)$ Prop. #3

Now, we can differentiate both sides of the equation:

$\frac{d}{dx}\{\ln [f(x)]\} = 3 \bullet \frac{d}{dx}\{\ln (x^2 - 3)\} - \frac{1}{6} \bullet \frac{d}{dx}\{\ln (3x + 5)\}$

$\frac{1}{f(x)} \bullet f'(x) = 3 \bullet \frac{1}{x^2 - 3} \bullet \frac{d}{dx}(x^2 - 3) - \frac{1}{6} \bullet \frac{1}{3x + 5} \bullet \frac{d}{dx}(3x + 5)$

$\frac{1}{f(x)} \bullet f'(x) = 3 \bullet \frac{1}{x^2 - 3} \bullet (2x) - \frac{1}{6} \bullet \frac{1}{3x + 5} \bullet (3)$

$\frac{1}{f(x)} \bullet f'(x) = \frac{6x}{x^2 - 3} - \frac{1}{2(3x + 5)}$

Now, we will solve for f '(x) by multiplying both sides by f(x):

$f'(x) = (\frac{6x}{x^2 - 3} - \frac{1}{2(3x + 5)}) \bullet f(x)$

But $f(x) = \frac{(x^2 - 3)^3}{\sqrt[6]{3x+5}}$. Thus, f '(x) becomes:

$f'(x) = (\frac{6x}{x^2 - 3} - \frac{1}{2(3x + 5)}) \frac{(x^2 - 3)^3}{\sqrt[6]{3x+5}}$

Ex. 24 Show that if $f(x) = e^x$, then $f'(x) = e^x$.

<u>Solution:</u>

We begin by taking the natural log of both sides:

ln [f(x)] = **ln** (e^x)

$\ln [f(x)] = x$ (inverse functions)

Now, differentiate both sides:

$\frac{d}{dx}\{\ln [f(x)]\} = \frac{d}{dx}\{x\}$

$\frac{1}{f(x)} \bullet f'(x) = 1$

Now, solve for f '(x) by multiplying by both sides by f(x):

$f'(x) = 1 \bullet f(x)$

$f'(x) = f(x)$ But, $f(x) = e^x$, thus

$f'(x) = e^x$.