Section 5.1 – Integration

Often in mathematics, a tool or process is developed and then later, the tool or process is used backwards. For example, when people are first introduced to fractions, they are taught how to reduce a fraction to lowest terms. Later, they reverse the process when adding or subtracting fractions with different denominators. Similarly, in Algebra, people are taught to "foil" the product of two binomials and then later reverse that process to factor trinomials. The same idea goes for Calculus. In the first four chapters of the book, we have been differentiating functions. Now, we are going to reverse that process. We can think of it as we are given the derivative of the function and we are trying to find the original function. This process is called antidifferentiation.

Ex. 1 Given the derivative below, find a possible function that yields that derivative:

a) $y' = 2x$ b) $y' = 0.5e^{0.5x}$

c) $y' = \frac{1}{x}$ for $x > 0$ d) $y' = 3x^2$

Solution:

a) If $y' = 2x$, then a possible function is $y = x^2$ since $\frac{d}{dx}[x^2] = 2x$. Other possible functions are $x^2 + 2$, $x^2 - \pi$, or $x^2 + c$ where c is a constant since the derivative of a constant is zero.

b) If $y' = 0.5e^{0.5x}$, then a possible function is $y = e^{0.5x}$ since $\frac{d}{dx}[e^{0.5x}] = 0.5e^{0.5x}$. Other possible functions are $e^{0.5x} + 1$, $e^{0.5x} - 3$, or $e^{0.5x} + c$ where c is a constant since the derivative of a constant is zero.

c) If $y' = \frac{1}{x}$ for $x > 0$, then a possible function is $y = \ln(x)$ since $\frac{d}{dx}[\ln(x)] = \frac{1}{x}$ for $x > 0$. Other possible functions are $\ln(x) + 3$, $\ln(x) - e$, or $\ln(x) + c$ where c is a constant since the derivative of a constant is zero.

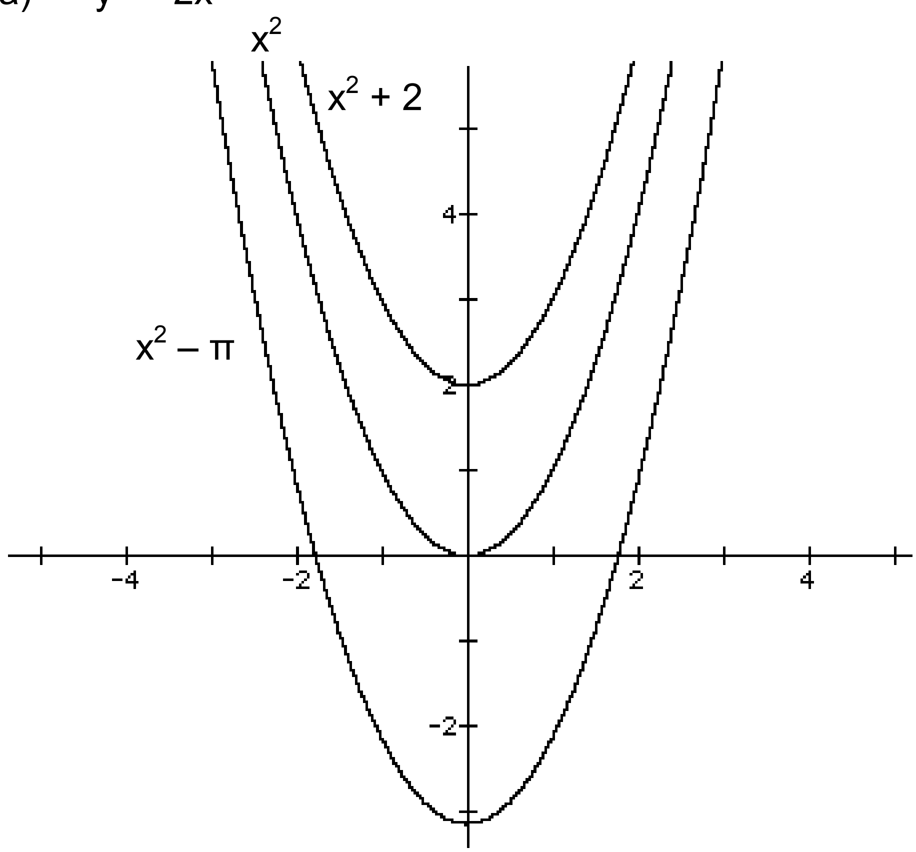
d) If $y' = 3x^2$, then a possible function is $y = x^3$ since $\frac{d}{dx}[x^3] = 3x^2$. Other possible functions are $x^3 + 1.5$, $x^3 - 0.5$, or $x^3 + c$ where c is a constant since the derivative of a constant is zero.

We can see from example #1 that each function has an infinite number of antiderivatives. We can think of it as there is a whole family of solutions, each differing only by a constant. Let's explore this a little further:

Ex. 2 Find and graph three different antiderivatives for the functions given in example #1:

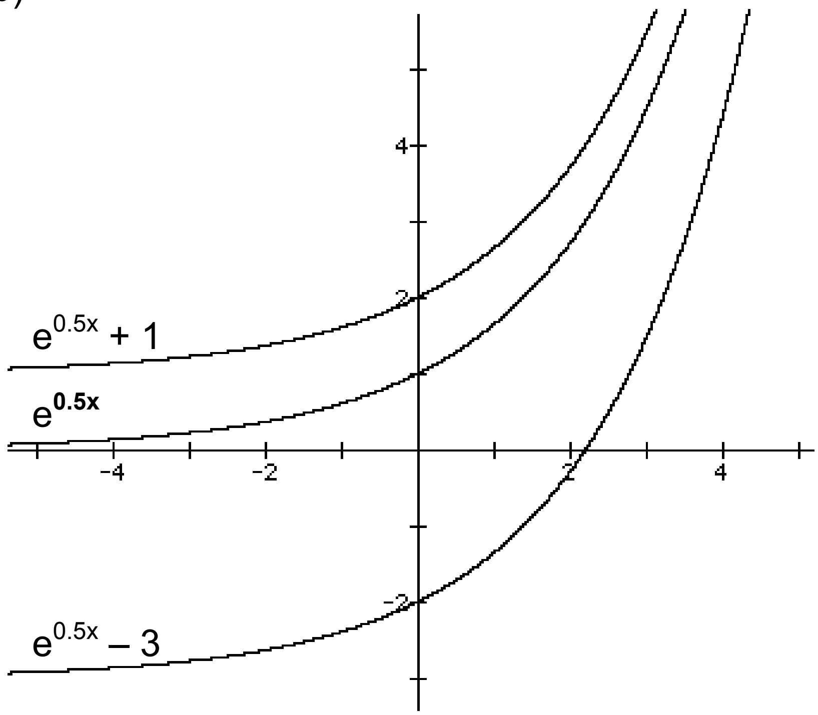
<u>Solution:</u>

a) $y' = 2x$



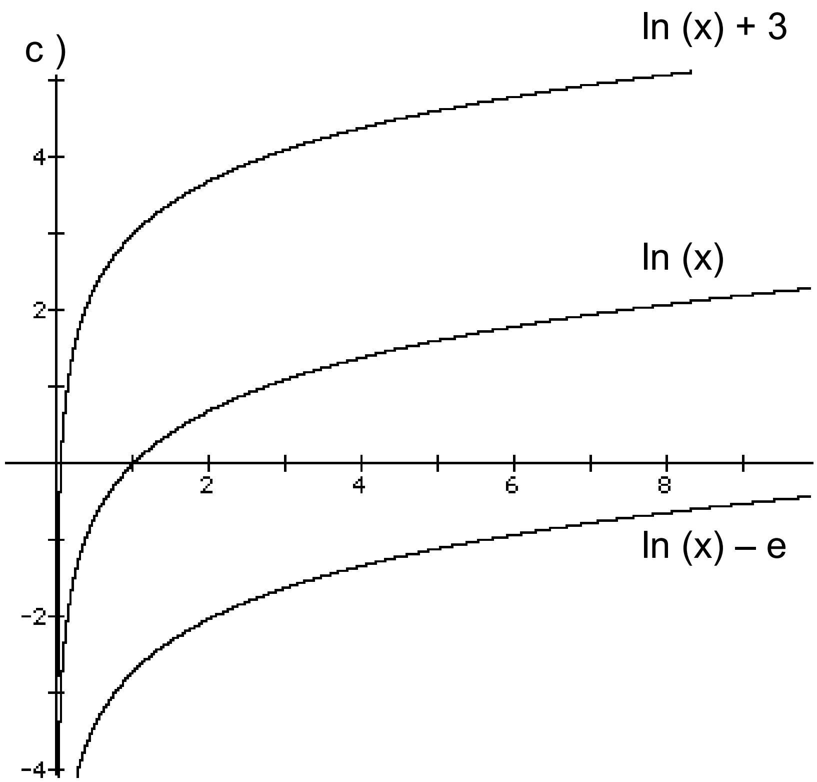
Notice that all three of these functions have the derivative of $y' = 2x$. If we examine the slope of the tangent line for any value x, it will have the same value on each of the curves.

b)



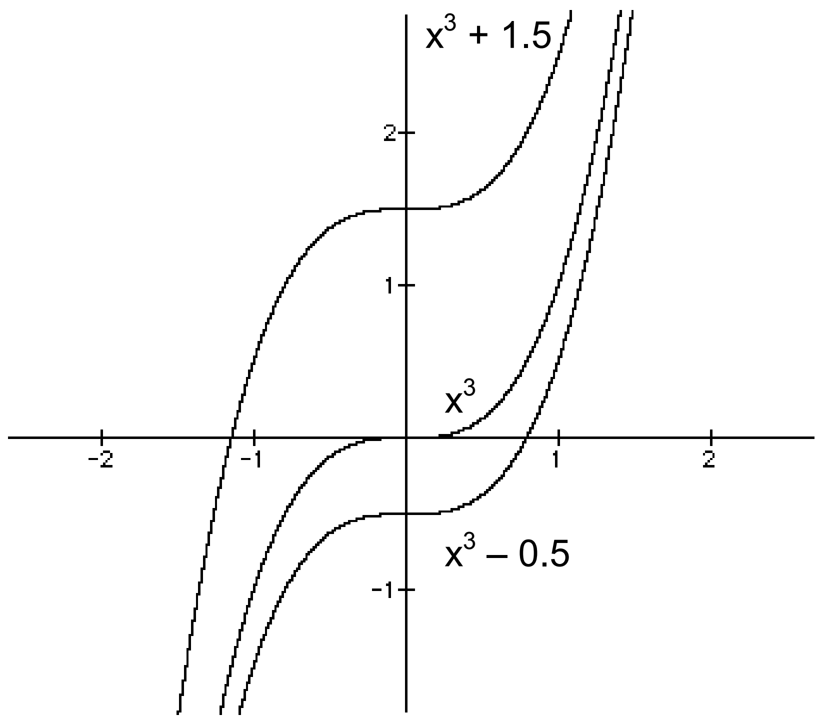
Notice that all three of these functions have the derivative of y ' = $0.5e^{0.5x}$. If we examine the slope of the tangent line for any value x, it will have the same value on each of the curves.

c)



Notice that all three of these functions have the derivative of y ' = $\frac{1}{x}$, x > 0. If we examine the slope of the tangent line for any value x, it will have the same value on each of the curves.

d)



Notice that all three of these functions have the derivative of y ' = $3x^2$. If we examine the slope of the tangent line for any value x, it will have the same value on each of the curves.

Now, let us give a more formal definition of the antiderivative:

Antiderivative

A function F(x) for which F '(x) = f(x) for all x in the domain of f is said to be the **antiderivative** of f(x).

Fundamental property of Antideriavtives

If F(x) and G(x) are two antiderivatives of a continuous function f(x), then G(x) = F(x) + c for some constant c.

Now we will introduce some new notation involving finding antiderivatives:

Notation for the Indefinite Integral:

The antiderivative of $3x^2$ with respect to x is $x^3 + c$.

$$\int 3x^2\,dx = x^3 + c \longleftarrow \quad \text{Constant of Integration}$$

Integral Sign — with respect to x

We will refer to antidifferentiation as integration from here on.

Rules for integration

Let k, n, and c be constants. Then the following holds:

1) $\int k\,dx = kx + c$

2) $\int x^n\,dx = \frac{x^{n+1}}{n+1} + c,\ n \neq -1$ (Power Rule for Integration)

3) $\int x^{-1}\,dx = \int \frac{1}{x}\,dx = \ln(|x|) + c,\ x \neq 0$

4) $\int e^{kx}\,dx = \frac{e^{kx}}{k} + c,\ k \neq 0$

5) $\int k \cdot f(x)\,dx = k\int f(x)\,dx$

6) $\int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx$ Integration term by term

Ex. 3 Integrate the following:

a) $\int \sqrt[3]{t}\,dt$ b) $\int (x^{1/2} + 3x^{4/3} + 6)\,dx$

Solution:

a) We will begin by writing $\sqrt[3]{t}$ as a power. Then we will integrate using the Power Rule.

$$\int \sqrt[3]{t}\,dt = \int t^{1/3}\,dt = \frac{t^{\frac{1}{3}+1}}{\frac{1}{3}+1} + c = \frac{t^{\frac{4}{3}}}{\frac{4}{3}} + c = \frac{3t^{\frac{4}{3}}}{4} + c$$

$$= \frac{3t\sqrt[3]{t}}{4} + c.$$

b) We will integrate term by term:

$$\int (x^{1/2} + 3x^{4/3} + 6)\, dx$$

$$= \int x^{1/2}\, dx + \int 3x^{4/3}\, dx + \int 6\, dx$$

$$= \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + \frac{3x^{\frac{4}{3}+1}}{\frac{4}{3}+1} + 6x + c = \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + \frac{3x^{\frac{7}{3}}}{\frac{7}{3}} + 6x + c$$

$$= \frac{2}{3}x^{3/2} + \frac{3}{7}\bullet 3x^{7/3} + 6x + c = \frac{2}{3}x^{3/2} + \frac{9}{7}x^{7/3} + 6x + c.$$

Integrate:

Ex. 4 $\int \left(\frac{1}{2y} - \frac{2}{y^5} + \frac{6}{\sqrt{y}}\right) dy.$

Solution:

We begin by integrating term by term and writing the terms as powers:

$$\int \frac{1}{2y} dy - \int \frac{2}{y^5}\, dy + \int \frac{6}{\sqrt{y}} dy$$

$$= \frac{1}{2}\int y^{-1}\, dy - 2\int y^{-5}\, dy + 6\int y^{-1/2}\, dy$$

$$= \frac{1}{2}\ln(|y|) - 2\bullet\frac{y^{-4}}{-4} + 6\bullet\frac{y^{\frac{1}{2}}}{\frac{1}{2}} + c$$

$$= \frac{1}{2}\ln(|y|) + \frac{1}{2y^4} + 12\sqrt{y} + c$$

Ex. 5 $\int (x - 3e^{0.2x})\, dx.$

Solution:

Again, we integrate term by term:

$$\int x\, dx - 3\int e^{0.2x}\, dx = \frac{x^2}{2} - 3\bullet\frac{e^{0.2x}}{0.2} + c$$

$$= \frac{x^2}{2} - 15e^{0.2x} + c.$$

Ex. 6 $\int e^{-0.06x}(1 + e^{0.12x})\, dx$

Solution:
We first need to simplify the expression we are trying to integrate and then integrate term by term:

$$\int e^{-0.06x}(1 + e^{0.12x})\,dx = \int (e^{-0.06x} + e^{-0.06x}e^{0.12x})\,dx$$

$$= \int (e^{-0.06x} + e^{0.06x})\,dx = \frac{e^{-0.06x}}{-0.06} + \frac{e^{0.06x}}{0.06} + c$$

But $\frac{1}{0.06} = \frac{100}{6} = \frac{50}{3}$, thus $\frac{e^{-0.06x}}{-0.06} + \frac{e^{0.06x}}{0.06} + c$

$= -\frac{50}{3}e^{-0.06x} + \frac{50}{3}e^{0.06x} + c.$

Sometimes we are given an initial condition or a point on the curve of the antiderivative. This allows us to solve for c. Let's take a look at an example

Ex. 7 Find a function whose tangent line has slope of $6x^2 - 8x$ for each value of x and whose graph passes through the point (1, 5).

Solution:
First, we integrate term by term to get the indefinite integral:

$$\int (6x^2 - 8x)\,dx = \int 6x^2\,dx - \int 8x\,dx = \frac{6x^3}{3} - \frac{8x^2}{2} + c$$

$= 2x^3 - 4x^2 + c$. Since the graph passes through the point (1, 5), this means that if we evaluate the function at x = 1, we will get 5 as our answer:

$2(1)^3 - 4(1)^2 + c = 5$
$2 - 4 + c = 5$
$-2 + c = 5$
$c = 7$

Thus, the function we are looking for is:
$f(x) = 2x^3 - 4x^2 + 7$.

Ex. 8 Find a function whose graph has a relative minimum at $x = -3$ and a relative maximum at $x = 2$.

Solution:
Since the function has a relative minimum at $x = -3$ and a relative maximum at $x = 2$, this means that $x = -3$ and $x = 2$ are critical values. Thus, y ' is zero when $x = -3$ and $x = 2$. In other words, $(x + 3)$ and $(x - 2)$

are factors of y '. A candidate for y ' is:

$y' = (x + 3)(x - 2) = x^2 + x - 6$

We need to check the second derivative to see if it gives us the correct relative extrema:

$y'' = \frac{d}{dx}[x^2 + x - 6] = 2x + 1.$

$y''\Big|_{x=2} = 2(2) + 1 = 5 > 0 \quad \cup$ - rel. min.

$y''\Big|_{x=-3} = 2(-3) + 1 = -5 < 0 \quad \cap$ - rel. max.

We have our signs backwards. To fix this problem, we take the opposite of our candidate for y ':

$y' = -(x^2 + x - 6) = -x^2 - x + 6$

Now, we can integrate term by term:

$\int (-x^2 - x + 6)\,dx = -\frac{x^3}{3} - \frac{x^2}{2} + 6x + c$

Ex. 9 An object is moving so that its velocity after t minutes is $v(t) = 5 + 8t + 3t^2$ meters per minute. How far does the object travel in during the second minute.

Solution:

If s(t) is the position function, the distance travel during the second minute will be s(2) – s(1). We will need to integrate v(t), evaluate it at t = 2 and t = 1 and then subtract:

$s(t) = \int (5 + 8t + 3t^2)\,dt = 5t + \frac{8t^2}{2} - \frac{3t^3}{3} + c$

$= 5t + 4t^2 - t^3 + c$. Evaluating s(t) at t = 2 and 1, we get:

$s(2) = 5(2) + 4(2)^2 - (2)^3 + c = 10 + 16 - 8 + c = 18 + c$

$s(1) = 5(1) + 4(1)^2 - (1)^3 + c = 5 + 4 - 1 + c = 8 + c.$

Thus,

$s(2) - s(1) = 18 + c - (8 + c) = 10.$

Hence, the object traveled 10 meters during the second minute.

Ex. 10 A retailer receives a shipment of 1400 pounds of coffee that will be used at constant rate of 200 pounds per day. If the cost of storing the coffee is 0.8 cents per pound per day, how much will the retailer have to pay in storage costs over the next seven days?

Solution:

Let t be the number of days and let s(t) be the total storage cost. Since the coffee is used at a constant rate of 200 pounds a day, the amount of coffee in storage after t days is 1400 – 200t. Since it costs 0.8 ¢ or \$0.008 per pound per day to store the coffee, the rate of change of the storage cost with respect to time is:

s '(t) = (cost per pound per day)•(number of pounds)

= 0.008(1400 – 200t) = 11.2 – 1.6t

Integrating yields:

$$s(t) = \int (11.2 - 1.6t)\,dt = 11.2t - 0.8t^2 + c$$

But, just as the shipment arrives (t = 0), no storage cost has been occurred (s(t) = 0). Using this fact, we find that:

$$s(0) = 11.2(0) - 0.8(0)^2 + c = 0$$
$$c = 0$$

Thus, $s(t) = 11.2t - 0.8t^2$

The total storage cost for 1 week is:

$s(7) = 11.2t - 0.8t^2 = 11.2(7) - 0.8(7)^2 = 39.2$

So, the storage costs will be \$39.20.

Ex. 11 An environmental study of a certain community suggests that t years from now, the level of carbon monoxide in the atmosphere will be changing at the rate of 0.05t + 0.2 parts per million per year. If the current level is 2.5 parts per million, what will the level be 5 years from now?

Solution:

We begin by integrating 0.05t + 0.2:

$$C(t) = \int (0.05t + 0.2)\,dt = \frac{0.05t^2}{2} + 0.2t + c$$

$= 0.025t^2 + 0.2t + c$. Currently, the level is 2.5 parts per million which means C(0) = 2.5. Plugging in and solving for c yields:

$C(0) = 0.025(0)^2 + 0.2(0) + c = 2.5$ or $c = 2.5$

Thus, $C(t) = 0.025t^2 + 0.2t + 2.5$.

Now, we can find C(5):

$C(5) = 0.025(5)^2 + 0.2(5) + 2.5 = 4.125$.

Five years from now, the carbon monoxide level will be 4.125 parts per million.