Section 5.2 – Area and The Definite Integral

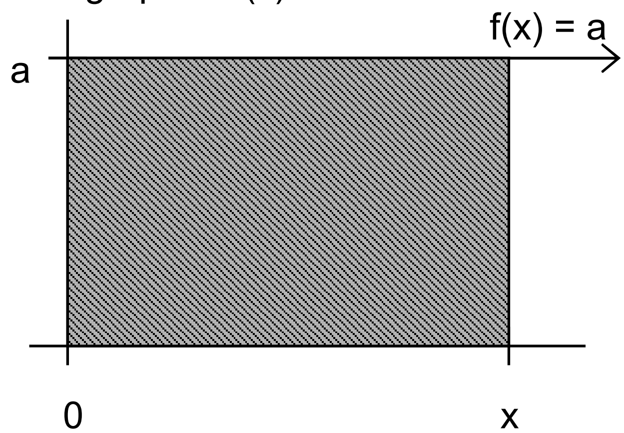
Before we begin our discussion of the definite integral, let's explore finding the area under different types of curves.

Given a function, find the area A(x) under the function from 0 to x:

Ex. 1 $f(x) = a$

Solution:

The graph of $f(x) = a$ is a horizontal line:



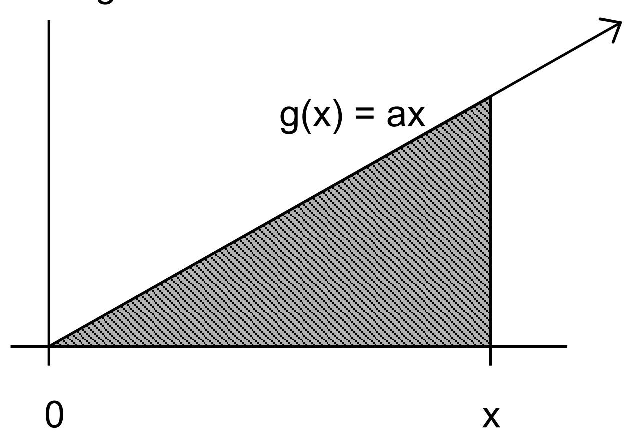
But, this is just a rectangle with the length equal to x and the width equal to a. So, the area is $x \cdot a$ or ax.

Hence, $A(x) = ax$.

Ex. 2 $g(x) = ax$

Solution:

The graph of $g(x)$ is a linear equation passing through the origin:



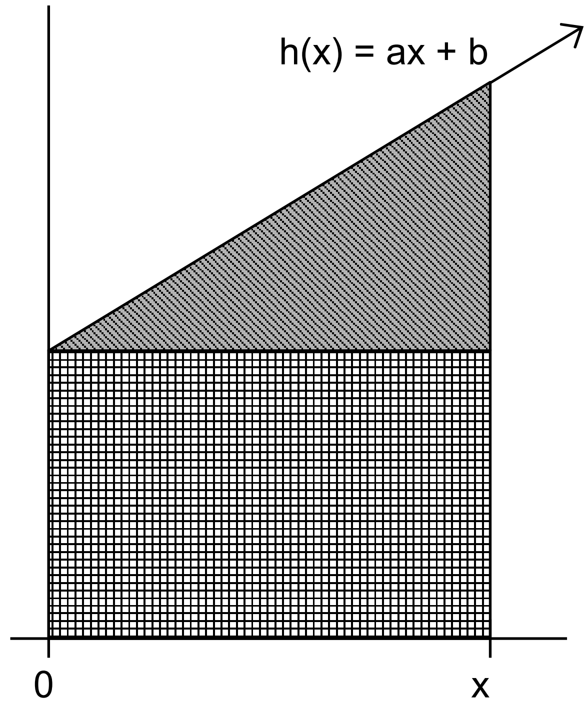
Notice that the area under the function is a triangle with a base of x and a height of ax. So, the area is $\frac{1}{2}bh$ or $\frac{1}{2}x \cdot ax = \frac{1}{2}ax^2$.

Hence, $A(x) = \frac{1}{2}ax^2$.

Ex. 2 $h(x) = ax + b$

Solution:

The graph of g(x) is a linear equation passing through the point (0, b):



Notice that the area under the function is a triangle with a base of x and a height of ax plus a rectangle with the length of x and a width of b. The area is the area of the triangle plus the the area of the rectangle. So the area is $\frac{1}{2}ax^2 + bx$.

Hence, $A(x) = \frac{1}{2}ax^2 + bx$.

Notice the area function is an antidervative of the original function. Now, let us explore the next example.

Ex. 4 The marginal cost (in dollars) for a certain commodity is $C'(x) = 2x + 1$ where x is the number of units. Find the increase in production cost when the number of units produced increases from 2 units to 5 units.

Solution:

We will need to integrate $C'(x)$, evaluate it at $x = 5$ and $x = 2$ and then subtract:

$$C(x) = \int (2x + 1)\,dx = x^2 + x + c$$

Evaluating C at $x = 5$ and $x = 2$ yields:

$C(5) = (5)^2 + (5) + c = 30 + c$

$C(2) = (2)^2 + (2) + c = 6 + c$

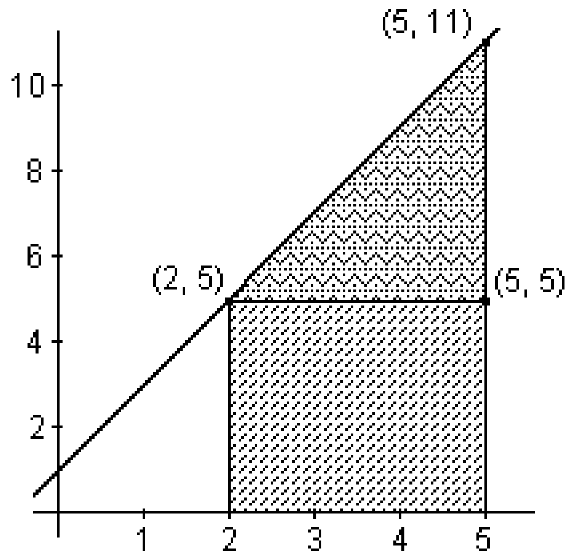
So, $C(5) - C(2) = 30 + c - (6 + c) = 30 + c - 6 - c = 24$.

Hence, the increase on production cost will be \$24.

Ex. 5 Find the area under the curve of C '(x) = 2x + 1, above the x - axis and between x = 2 and x = 5.

Solution:

We begin by drawing the region we are trying to find the area of:



Notice that the area of the region can be split into a rectangle and a triangle. The rectangle is 3 units long and \$5 per unit high so that its area is \$15. The length of the base of the triangle is 3 units and it is \$6 per unit high so that its area is \$9. The sum of the two areas is \$24 which is the same we got in Ex. 4.

This is not a coincidence. For any continuous function f(x) where f(x) ≥ 0 for all x in a specified interval [a, b], the $\int f(x)\,dx$ evaluated at b minus $\int f(x)\,dx$ evaluated at a will correspond to area under the curve from a to b.

Area under a curve

If f(x) is continuous and f(x) ≥ 0 on [a, b], then the region under the curve y = f(x) above the interval [a, b] has area of:

$$A = \int_a^b f(x)\,dx$$

a is referred to as the lower limit of integration and b is referred as the upper limit of integration.

Fundamental Theorem of Calculus

If f(x) is continuous on [a, b] and F(x) is <u>any</u> antiderivative of f(x), then

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a)$$

The reason why we can use any antiderivative is because when take F(b) – F(a), the constant c in the antiderivative will subtract out. Since we can choose to use any antiderivative of f(x), then we will choose to use the antiderivative where c is 0.

Integrate the following:

Ex. 6 $\int_1^9 (\sqrt{t} - \frac{4}{\sqrt{t}})\, dt$

Solution:

$\int_1^9 (\sqrt{t} - \frac{4}{\sqrt{t}})\, dt = \int_1^9 (t^{1/2} - 4t^{-1/2})\, dt$

$= \int_1^9 t^{1/2} dt - 4\int_1^9 t^{-1/2} dt = [\frac{t^{3/2}}{3/2} - 4\frac{t^{1/2}}{1/2}] \Big|_1^9$

$= [\frac{2}{3}t^{3/2} - 8t^{1/2}] \Big|_1^9 = [\frac{2}{3}(9)^{3/2} - 8(9)^{1/2}] - [\frac{2}{3}(1)^{3/2} - 8(1)^{1/2}]$

$= [\frac{2}{3}(27) - 8(3)] - [\frac{2}{3}(1) - 8(1)] = [18 - 24] - [\frac{2}{3} - 8]$

$= -6 + 7\frac{1}{3} = 1\frac{1}{3}$.

Ex. 7 $\int_0^3 (3x^2 - 6x + 5)\, dx$

Solution:

$\int_0^3 (3x^2 - 6x + 5)\, dx = [x^3 - 3x^2 + 5x] \Big|_0^3$

$= [(3)^3 - 3(3)^2 + 5(3)] - [(0)^3 - 3(0)^2 + 5(0)]$

$= [27 - 27 + 15] - [0] = 15$.

Ex. 8 $\int_1^5 x^2(x - 3)\, dx$

Solution:

We begin by distributing the x^2 and then integrating:

$\int_1^5 x^2(x - 3)\, dx = \int_1^5 (x^3 - 3x^2)\, dx = (\frac{x^4}{4} - x^3) \Big|_1^5$

$= (\frac{(5)^4}{4} - (5)^3) - (\frac{(1)^4}{4} - (1)^3) = (\frac{625}{4} - 125) - (\frac{1}{4} - 1)$

$= \frac{625}{4} - 125 - \frac{1}{4} + 1 = \frac{624}{4} - 124 = 156 - 124 = 32$.

Ex. 9 $\int_1^4 (\frac{1}{x} + e^x)\,dx$

Solution:

We begin by integrating term by term:

$$\int_1^4 (\frac{1}{x} + e^x)\,dx = \int_1^4 \frac{1}{x}\,dx + \int_1^4 e^x\,dx = [\ln|x| + e^x]\Big|_1^4$$

$= [\ln|4| + e^4] – [\ln|1| + e^1] = \ln(4) + e^4 – [0 + e]$

$= \ln(4) + e^4 – e.$

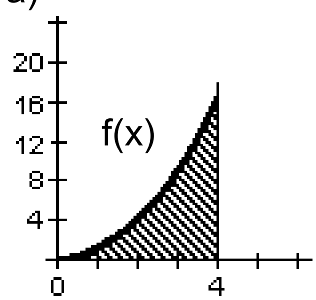
Ex. 10 Find the area under

a) $f(x) = x^2$ on the interval [0, 4].

b) $g(x) = – x^2$ on the interval [0, 4]

Solution:

a)

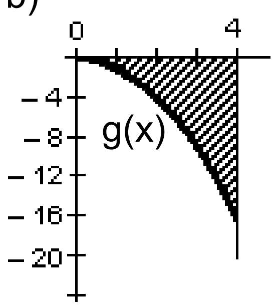


The area under the curve is equal the integral of x^2 from x = 0 to x = 4.

$$A = \int_0^4 x^2\,dx = \frac{x^3}{3}\Big|_0^4$$

$$= \frac{4^3}{3} – \frac{0^3}{3} = \frac{64}{3} – 0 = \frac{64}{3} \text{ sq. units.}$$

b)



By symmetry, the area under the should be equal to the result in part a. Let's see what happens when we integrate $– x^2$ from x = 0 to x = 4:

$$A = \int_0^4 – x^2\,dx = – \frac{x^3}{3}\Big|_0^4$$

$$= – \frac{4^3}{3} – \left(– \frac{0^3}{3}\right) = – \frac{64}{3} + 0 = – \frac{64}{3}.$$

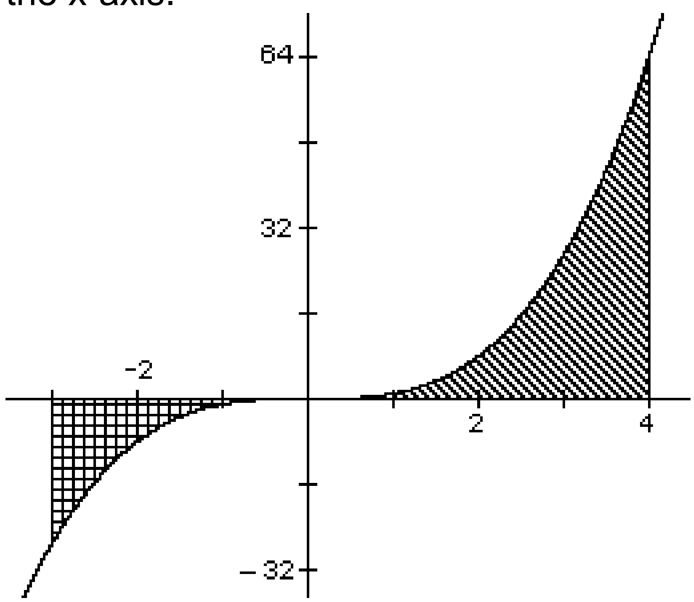
So, the area is $\frac{64}{3}$ sq. units.

Notice that for the negative value functions, the definite integral gives us a negative result.

Ex. 11 Evaluate $\int_{-3}^{4} x^3\,dx$ and interpret the results.

<u>Solution:</u>

If we examine a picture of the graph, we can see that part of the area is above the x-axis and part of it is below the x-axis.



Thus, if we integrate x^3 from – 3 to 0, the result should be negative and if we integrate from 0 to 4, the result should be positive:

$$\int_{-3}^{0} x^3\,dx = \frac{x^4}{4}\Big|_{-3}^{0}$$

$$= \frac{0^4}{4} - \frac{(-3)^4}{4} = -\frac{81}{4}.$$

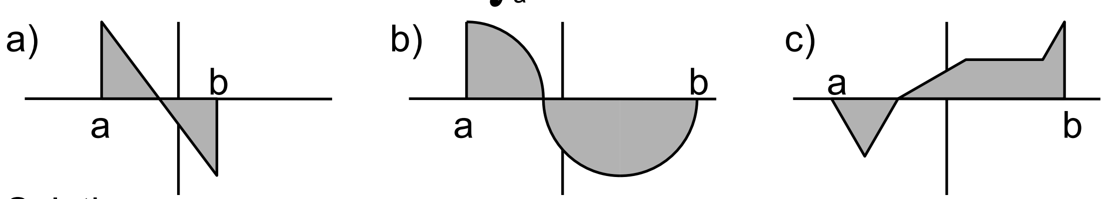
$$\int_{0}^{4} x^3\,dx = \frac{x^4}{4}\Big|_{0}^{4}$$

$$= \frac{4^4}{4} - \frac{0^4}{4} = 64.$$

But $\int_{-3}^{4} x^3\,dx = \frac{x^4}{4}\Big|_{-3}^{4} = \frac{4^4}{4} - \frac{(-3)^4}{4} = 64 - \frac{81}{4} = 43.75.$

Notice that the definite integral is equal to the area above the x-axis minus the area below the x-axis.

Ex. 12 Given the graph of f, is $\int_{a}^{b} f(x)\,dx$ positive, negative, or 0?



<u>Solution:</u>

a) Since the areas are equal, then $\int_{a}^{b} f(x)\,dx = 0$

b) There is more area below the x-axis, so $\int_{a}^{b} f(x)\,dx < 0$.

c) There is more area above the x-axis, so $\int_{a}^{b} f(x)\,dx > 0$.