Section 5.5 – Integration by Substitution

Many times in mathematics, we need to make a substitution in a problem in order to solve it. By using substitution, we get a problem that is easier to work. One classic case is when we are factoring. Let's look at an example.

Ex. 1 Factor $6(x + 3y)^2 – (x + 3y) – 1$.

Solution:

I) We begin by recognizing that this is a quadratic-like expression, i.e.,

$$6(\quad)^2 – (\quad) – 1$$

II) Thus, if we let $u = (x + 3y)$, then the expression becomes:

$$6u^2 – u – 1$$

III) Now, we can factor this problem:

$$6u^2 – u – 1 = (3u + 1)(2u – 1)$$

IV) Finally, we substitute $u = (x + 3y)$ back into the problem and simplify:

$$(3u + 1)(2u – 1) = (3(x + 3y) + 1)(2(x + 3y) – 1)$$
$$= (3x + 9y + 1)(2x + 6y – 1)$$

Notice that there were four steps involved in working this problem:

I) Identify the pattern that you will use.
II) Make a substitution.
III) Work the new problem.
IV) Substitute the original expression into your answer and simplify.

In calculus, when we are integrating, we will make a substitution to make the problem easier to work. There are three basic patterns that we are looking for:

1) $\int u^n\,du = \frac{u^{n+1}}{n+1} + c, \quad n \neq -1$

2) $\int u^{-1}\,du = \int \frac{1}{u}\,du = \ln|u| + c$

3) $\int e^u\,du = e^u + c$

To complete the substitution, we will need to find du. This is done by differentiating the substitution and solving for du. For example, if $u = 3x^2 + 6$, then $\frac{du}{dx} = 6x$. If you treat $\frac{du}{dx}$ as a fraction and multiply both sides by dx, that will give you du = 6x dx. We then have to substitute in the du as well. To see how all this works, let's look at some examples:

Integrate the following:

Ex. 2 $\int_0^2 6x(3x^2 + 6)^5\, dx$

Solution:

We begin by identifying which pattern we need to use. Since there is no denominator, then we should not consider #2 and since e is not involved, we can eliminate #3. Thus, we want use pattern #1. Hence, we need to make substitution in order that we have the integral written as a power.

Let $u = 3x^2 + 6$

$\frac{du}{dx} = 6x \quad \Rightarrow \quad du = 6x\, dx$

Substituting, we get:

$$\int_0^2 \mathbf{6x}(\underline{3x^2 + 6})^5\, \mathbf{dx} = \int_{x=0}^{x=2} (\underline{u})^5\, \mathbf{du} = \int_{x=0}^{x=2} u^5\, du.$$

Now, we can integrate:

$$\int_{x=0}^{x=2} u^5\, du = \frac{u^6}{6}\Big|_{x=0}^{x=2} \text{ and then substitute } u = 3x^2 + 6:$$

$$= \frac{u^6}{6}\Big|_{x=0}^{x=2} = \frac{(3x^2+6)^6}{6}\Big|_0^2 = \frac{(3(2)^2+6)^6}{6} - \frac{(3(0)^2+6)^6}{6}$$

$$= \frac{(18+6)^6}{6} - \frac{(6)^6}{6} = 31{,}850{,}496 - 7776 = 31{,}842{,}720$$

Ex. 3 $\int \frac{9x^2}{3x^3 - 5}\, dx$

Solution:

Since there is no e involved and the integral cannot be written as a power, the only choice is pattern #2:

Let $u = 3x^3 - 5$

$\frac{du}{dx} = 9x^2 \quad \Rightarrow \quad du = 9x^2\, dx$

Substituting, we get:

$$\int \frac{9x^2}{3x^3-5}\,dx = \int \frac{1}{3x^3-5}\bullet 9x^2\,dx = \int \frac{1}{u}\,du = \ln|u| + c$$

Replacing u by $3x^3 - 5$, we get:

$\ln|u| + c = \ln|3x^3 - 5| + c.$

Ex. 4 $\int_0^1 2x\, e^{x^2-1}\,dx$

Solution:

Since the integral involves e, we want to use pattern #3.

Let $u = x^2 - 1$

$\frac{du}{dx} = 2x \quad \Rightarrow \quad du = 2x\,dx$

Substituting, we get:

$$\int_0^1 2x\, e^{x^2-1}\,dx = \int_0^1 e^{x^2-1}\, 2x\,dx = \int_{x=0}^{x=1} e^u\,du = e^u \Big|_{x=0}^{x=1}$$

Replacing u with $x^2 - 1$ yields:

$$e^u \Big|_{x=0}^{x=1} = e^{x^2-1} \Big|_0^1 = e^{1-1} - e^{0-1} = e^0 - e^{-1}$$

$= 1 - e^{-1} \approx 0.63212$

Ex. 5 $\int \sqrt{7x-9}\,dx$

Solution:

There is no denominator or e involved in the problem so we will need to use pattern #1.

Let $u = 7x - 9$

$du = 7\,dx$

Since the integral only has a dx in it as oppose to 7 dx, we will need to divide both sides by 7.

$\frac{1}{7}\,du = dx$

Substituting, we get:

$$\int \sqrt{7x-9}\,dx = \int \sqrt{u}\,\frac{1}{7}\,du = \frac{1}{7}\int u^{1/2}\,du = \frac{1}{7}\bullet\frac{u^{3/2}}{\frac{3}{2}} + c$$

$$= \frac{1}{7}\bullet\frac{2}{3}u^{3/2} + c = \frac{2}{21}u\sqrt{u} + c$$

Replacing u with 7x – 9 yields:

$$\frac{2}{21}u\sqrt{u} + c = \frac{2}{21}(7x-9)\sqrt{7x-9} + c.$$

Ex. 6 $\int \frac{6y}{3y+4}\,dy$

<u>Solution:</u>

We have y-terms in the top and bottom so it hard to determine what kind of substitution we should try to perform. We can try dividing first to see if that helps:

$$\begin{array}{r} 2 - \frac{8}{3y+4} \\ 3y + 4 \overline{) 6y + 0} \\ \underline{-6y - 8} \\ -8 \end{array}$$

Thus, $\int \frac{6y}{3y+4}\,dy = \int \left(2 - \frac{8}{3y+4}\right) dy = 2y - 8\int \frac{1}{3y+4}\,dy$

Now, we can see that we need to use pattern #2.

Let $u = 3y + 4$

$du = 3\,dy$ solving for dy yields:

$\frac{du}{3} = dy$

Substituting, we get:

$2y - 8\int \frac{1}{3y+4}\,dy = 2y - 8\int \frac{1}{u}\frac{du}{3} = 2y - \frac{8}{3}\int \frac{1}{u}\,du$

$= 2y - \frac{8}{3}\ln|u| + c$. Replacing u by 3y + 4, we get:

$2y - \frac{8}{3}\ln|u| + c = 2y - \frac{8}{3}\ln|3y + 4| + c$.

Ex. 7 $\int \frac{x^5}{e^{1-x^6}}\,dx$

<u>Solution:</u>

Since the problem involves e, we will probably need to use pattern #3. Before we can do that, we need to bring the exponential into the numerator:

$\int \frac{x^5}{e^{1-x^6}}\,dx = \int x^5 e^{-(1-x^6)}\,dx = \int x^5 e^{x^6-1}\,dx$

Now, we can use pattern #3:

Let $u = x^6 - 1$

$du = 6x^5\,dx$ solving for $x^5\,dx$ yields:

$\frac{du}{6} = x^5\,dx$

Substituting, we get:

$$\int x^5 e^{x^6-1}\,dx = \int e^u \frac{du}{6} = \frac{1}{6}\int e^u\,du = \frac{1}{6}e^u + c$$

Replacing u by $x^6 - 1$, we get:

$$\frac{1}{6}e^u + c = \frac{1}{6}e^{x^6-1} + c.$$

Ex. 8 $\displaystyle\int_0^1 \frac{10x^3-5x}{\sqrt{x^4-x^2+6}}\,dx$

Solution:

First, let's factor 5 out of the numerator and bring it in front of the integral sign:

$$\int_0^1 \frac{10x^3-5x}{\sqrt{x^4-x^2+6}}\,dx = 5\int_0^1 \frac{2x^3-x}{\sqrt{x^4-x^2+6}}\,dx$$

If we try using pattern #2, we would end up with a big mess:

Let $u = (x^4 - x^2 + 6)^{1/2}$

$du = \frac{1}{2}(x^4 - x^2 + 6)^{-1/2}(4x^3 - 2x)\,dx$

Since this is more than we bargained for, we need to try a different pattern. Since pattern #1 is the only other feasible alternative, let's write the square root in the denominator as a power:

$$5\int_0^1 \frac{2x^3-x}{\sqrt{x^4-x^2+6}}\,dx = 5\int_0^1 (x^4-x^2+6)^{-1/2}[2x^3-x]\,dx$$

Let $u = x^4 - x^2 + 6$

$du = (4x^3 - 2x)\,dx$ solving for $(2x^3 - x)\,dx$ yields:

$\frac{du}{2} = (2x^3 - x)\,dx$

Substituting in, we get :

$$5\int_0^1 (x^4-x^2+6)^{-1/2}[2x^3-x]\,dx = 5\int_{x=0}^{x=1} u^{-1/2}\frac{du}{2}$$

$$= \frac{5}{2}\int_{x=0}^{x=1} u^{-1/2}\,du = \frac{5}{2}\frac{u^{1/2}}{\frac{1}{2}}\Bigg|_{x=0}^{x=1} = \frac{5}{2}\cdot 2\,u^{1/2}\Bigg|_{x=0}^{x=1} = 5u^{1/2}\Bigg|_{x=0}^{x=1}$$

Replacing u by $x^4 - x^2 + 6$, we obtain:

$$5u^{1/2}\Bigg|_{x=0}^{x=1} = 5(x^4-x^2+6)^{1/2}\Bigg|_0^1$$

$$= 5\sqrt{1^4-1^2+6} - 5\sqrt{0^4-0^2+6} = 5\sqrt{6} - 5\sqrt{6} = 0$$

Ex. 9 $\int \frac{e^{\sqrt{x}}}{(e^{\sqrt{x}} - 1)\sqrt{x}}\,dx$

<u>Solution:</u>

Since e is involved, one might try u = $\sqrt{x}$. But du = $\frac{1}{2\sqrt{x}}$ dx or 2 du = $\frac{1}{\sqrt{x}}$. If we substitute this into the integral, we do not get something that is easy to integrate:

$$\int \frac{e^{\sqrt{x}}}{(e^{\sqrt{x}} - 1)\sqrt{x}}\,dx = \int \frac{e^u}{e^u - 1}\,2\,du = 2\int \frac{e^u}{e^u - 1}\,du$$

Thus, pattern #3 does not work. So, we should try something different. Pattern #2 is the only other feasible choice in this problem:

Let u = $e^{\sqrt{x}} - 1$

du = $e^{\sqrt{x}} \bullet \frac{d}{dx}[\sqrt{x}\,] = e^{\sqrt{x}} \bullet \frac{1}{2\sqrt{x}}\,dx$ (multiply by 2)

2 du = $e^{\sqrt{x}} \bullet \frac{1}{\sqrt{x}}\,dx$

Substituting in, we get:

$$\int \frac{e^{\sqrt{x}}}{(e^{\sqrt{x}} - 1)\sqrt{x}}\,dx = \int \frac{1}{(e^{\sqrt{x}} - 1)} \bullet e^{\sqrt{x}} \bullet \frac{1}{\sqrt{x}}\,dx = \int \frac{1}{u}\,2\,du$$

$$= 2\int \frac{1}{u}\,du = 2\ln|u| + c$$

Replacing u by $e^{\sqrt{x}} - 1$, we obtain:

$2\ln|u| + c = 2\ln|e^{\sqrt{x}} - 1| + c.$

Ex. 10 $\int [(x - 4)^9 + 3(x - 4)^3 + 7]\,dx$

<u>Solution:</u>

Since pattern #2 or #3 does not apply in this situation, we want to try to use pattern #1:

Let u = (x – 4)

du = dx

Substituting in, we get:

$$\int [(x-4)^9 + 3(x-4)^3 + 7]\, dx = \int [u^9 + 3u^3 + 7]\, du$$

$$= \frac{u^{10}}{10} + \frac{3u^4}{4} + 7u + c$$

Replacing u by (x – 4), we obtain:

$$\frac{u^{10}}{10} + \frac{3u^4}{4} + 7u + c = \frac{(x-4)^{10}}{10} + \frac{3(x-4)^4}{4} + 7(x-4) + c.$$

Ex. 11 $\int \frac{1}{x[\ln(x)]^5}\, dx$

Solution:

Pattern #2 will not work since using $u = x\,[\ln(x)]^5$ would create a mess for du. Thus, the only feasible option is pattern #1. Let's write $[\ln(x)]^5$ in the denominator as a power:

$$\int \frac{1}{x[\ln(x)]^5}\, dx = \int [\ln(x)]^{-5} \frac{1}{x}\, dx$$

Let u = [ln (x)]

$du = \frac{1}{x}\, dx$

Substituting, we get:

$$\int [\ln(x)]^{-5} \frac{1}{x}\, dx = \int u^{-5}\, du = \frac{u^{-4}}{-4} + c = -\frac{1}{4u^4} + c$$

Replacing u by [ln (x)], we obtain:

$$-\frac{1}{4u^4} + c = -\frac{1}{4[\ln(x)]^4} + c$$

Ex. 12 When a machine is t years old, the rate at which its value is changing is $-960e^{-t/5}$ dollars per year. If the machine was bought new for $5,000, how much will it be worth 10 years later?

Solution:

We need to begin by integrating $-960e^{-t/5}$ with respect to t to find the function that will give us the value of the machine:

$$V(t) = \int -960e^{-t/5}\, dt = -960 \int e^{-t/5}\, dt$$

Let $u = -t/5$

$du = -\,dt/5$ solving for dt yields:
$-5\,du = dt$

Substituting, we get:

$V(t) = -960 \int e^{-t/5}\,dt = -960 \int e^{u}\,(-5)du$

$= 4800 \int e^{u}\,du = 4800e^{u} + c$

Replacing u by – t/5, we obtain:

$4800e^{u} + c = 4800e^{-t/5} + c$

When the machine was new (t = 0), then the value V(t) was \$5000. Plugging and solving for c yields:

$V(0) = 4800e^{-(0)/5} + c = 5000$
$4800e^{0} + c = 5000$
$4800 + c = 5000$
$c = 200$

Thus, $V(t) = 4800e^{-t/5} + c = 4800e^{-t/5} + 200$.

Ten years later corresponds to t = 10. Evaluating V(t) at 10 yields:

$V(10) = 4800e^{-t/5} + 200 = 4800e^{-(10)/5} + 200$
$= 4800e^{-2} + 200 \approx \$849.61$.

Hence, the machine was worth \$849.61 ten years later.