Section 5.7 – Integration using Tables

We have seen that integration can be much more difficult than differentiation. In fact, some integrals are beyond the scope of this course to calculate. In those cases, we will need to use a table of integrals. Think of the table of integrals as a formula sheet for computing integrals. The key is to match the integral to the correct formula in the table. Sometimes, you might have to do some substitution or integration by parts first before you can use a formula from the table of integrals. Let's examine the table that we will be using. The first five formulas should look very familiar.

Table #1

1. $\int u^n \, du = \frac{u^{n+1}}{n+1} + C$, where $n \neq -1$

2. $\int \frac{du}{u} = \ln|u| + C$

3. $\int e^u \, du = e^u + C$

4. $\int u \, dv = uv - \int v \, du$

5. $\int e^{kx} \, dx = \frac{e^{kx}}{k} + C$

6. $\int a^x \, dx = \frac{a^x}{\ln(a)} + C$, where $a > 0$ and $a \neq 1$

7. $\int xe^{ax} \, dx = \frac{1}{a^2} \bullet e^{ax}(ax - 1) + C$

8. $\int x^n e^{ax} \, dx = \frac{x^n e^{ax}}{a} - \frac{n}{a} \int x^{n-1} e^{ax} \, dx$

9. $\int \ln(x) \, dx = x \ln(x) - x + C$

10. $\int \log_a(x)\,dx = x\log_a(x) - \frac{x}{\ln(a)} + C$

11. $\int [\ln(x)]^n\,dx = x[\ln(x)]^n - n\int [\ln(x)]^{n-1}\,dx$, where $n \neq -1$

12. $\int x^n \ln(x)\,dx = x^{n+1}\left(\frac{\ln(x)}{n+1} - \frac{1}{(n+1)^2}\right) + C$, where $n \neq -1$

13. $\int \frac{dx}{\sqrt{x^2+a^2}} = \ln\left|x + \sqrt{x^2+a^2}\right| + C$

14. $\int \frac{dx}{\sqrt{x^2-a^2}} = \ln\left|x + \sqrt{x^2-a^2}\right| + C$

15. $\int \frac{dx}{x\sqrt{a^2+x^2}} = -\frac{1}{a}\ln\left|\frac{a+\sqrt{a^2+x^2}}{x}\right| + C$

16. $\int \frac{dx}{x\sqrt{a^2-x^2}} = -\frac{1}{a}\ln\left|\frac{a+\sqrt{a^2-x^2}}{x}\right| + C$

17. $\int \frac{x\,dx}{\sqrt{a+bx}} = \frac{2}{3b^2}(bx-2a)\sqrt{a+bx} + C$

18. $\int \frac{x^2\,dx}{\sqrt{a+bx}} = \frac{2}{15b^3}(3b^2x^2 - 4abx + 8a^2)\sqrt{a+bx} + C$

19. $\int \sqrt{x^2 \pm a^2}\,dx = \frac{1}{2}\left(x\sqrt{x^2 \pm a^2} \pm a^2\ln\left|x + \sqrt{x^2 \pm a^2}\right|\right) + C$

20. $\int \frac{\sqrt{a^2-x^2}}{x}\,dx = \sqrt{a^2-x^2} - a\ln\left|\frac{a+\sqrt{a^2-x^2}}{x}\right| + C$

21. $\int x\sqrt{a+bx}\,dx = \frac{2}{15b^2}(3bx-2a)(a+bx)^{3/2} + C$

22. $\int x^2\sqrt{a+bx}\,dx = \frac{2}{105b^3}(15b^2x^2 - 12abx + 8a^2)(a+bx)^{3/2} + C$

23. $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C$

24. $\displaystyle\int \frac{dx}{a^2 - x^2} = \frac{1}{2a}\ln\left|\frac{x+a}{x-a}\right| + C$

25. $\displaystyle\int \frac{x}{ax+b}\,dx = \frac{x}{a} - \frac{b}{a^2}\ln|ax+b| + C$

26. $\displaystyle\int \frac{x}{(ax+b)^2}\,dx = \frac{b}{a^2(ax+b)} + \frac{1}{a^2}\ln|ax+b| + C$

27. $\displaystyle\int \frac{dx}{x(ax+b)} = \frac{1}{b}\ln\left|\frac{x}{ax+b}\right| + C$

28. $\displaystyle\int \frac{dx}{x(ax+b)^2} = \frac{1}{b(ax+b)} + \frac{1}{b^2}\ln\left|\frac{x}{ax+b}\right| + C$

Evaluate the following integrals:

Ex. 1 $\displaystyle\int r\sqrt{6+5r}\,dr$

Solution:

This integral matches integral #21 in our table:

$$\int x\sqrt{a+bx}\,dx = \frac{2}{15b^2}(3bx - 2a)(a+bx)^{3/2} + C$$

Our variable of integration is r instead of x, but otherwise it fits the pattern. The value of a = 6 and b = 5, so we get:

$$\int r\sqrt{6+5r}\,dr$$

$$= \frac{2}{15(5)^2}(3(5)r - 2(6))((6) + (5)r)^{3/2} + C$$

$$= \frac{2}{375}(15r - 12)(6 + 5r)^{3/2} + C$$

$$= \frac{2}{375}(15r - 12)(6 + 5r)\sqrt{6+5r} + C$$

Ex. 2 $\displaystyle\int_0^2 \frac{5\,dx}{4x^2 - 36}$

Solution:

This integral almost matches integral #23. We need to get rid of the 5 in front of the dx and the 4 in front of x^2 term in the denominator. We can factor out a common factor of four out of the denominator and pull the constant $\frac{5}{4}$ in front of the integral:

$$\int_0^2 \frac{5\,dx}{4x^2 - 36} = \int_0^2 \frac{5\,dx}{4(x^2 - 9)} = \frac{5}{4}\int_0^2 \frac{dx}{x^2 - 9}.$$

Now, it matches integral #23:

$$\int \frac{dx}{x^2 - a^2} = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C$$

The value of a = 3 so we get:

$$\frac{5}{4}\int \frac{dx}{x^2 - 9} = \frac{5}{4}\bullet\frac{1}{2(3)}\ln\left|\frac{x-3}{x+3}\right| + C = \frac{5}{24}\ln\left|\frac{x-3}{x+3}\right| + C$$

Thus, $\frac{5}{4}\int_0^2 \frac{dx}{x^2 - 9} = \frac{5}{24}\ln\left|\frac{x-3}{x+3}\right|\Big|_0^2$

$$= \frac{5}{24}\ln\left|\frac{2-3}{2+3}\right| - \frac{5}{24}\ln\left|\frac{0-3}{0+3}\right| = \frac{5}{24}\ln(0.2) - \frac{5}{24}\ln(1)$$

$$= \frac{5}{24}\ln(0.2) \approx -0.3353$$

Ex. 3 $\quad \int_1^3 x^2e^{0.4x}\,dx$

Solution:

This integral matches integral #8:

$$\int x^ne^{ax}\,dx = \frac{x^ne^{ax}}{a} - \frac{n}{a}\int x^{n-1}e^{ax}\,dx$$

The value of a = 0.4 and n = 2, so we get:

$$\int x^2e^{0.4x}\,dx = \frac{x^2e^{0.4x}}{0.4} - \frac{2}{0.4}\int xe^{0.4x}\,dx$$

$$= 2.5x^2e^{0.4x} - 5\int xe^{0.4x}\,dx$$

But $\int xe^{0.4x}\,dx$ matches integral #7:

$$\int xe^{ax}\,dx = \frac{1}{a^2}\bullet e^{ax}(ax - 1) + C$$

The value of a = 0.4, so we get:

$$2.5x^2e^{0.4x} - 5\int xe^{0.4x}\,dx$$

$$= 2.5x^2e^{0.4x} - 5\left(\frac{1}{(0.4)^2}\bullet e^{0.4x}(0.4x - 1)\right) + C$$

$= 2.5x^2e^{0.4x} – 5(6.25e^{0.4x}(0.4x – 1)) + C$
$= 2.5x^2e^{0.4x} – 5(2.5xe^{0.4x} – 6.25e^{0.4x}) + C$
$= 2.5x^2e^{0.4x} – 12.5xe^{0.4x} + 31.25e^{0.4x} + C$

Hence, $\int_1^3 x^2e^{0.4x}\,dx$

$= [2.5x^2e^{0.4x} – 12.5xe^{0.4x} + 31.25e^{0.4x}] \Big|_1^3$
$= [2.5(3)^2e^{0.4(3)} – 12.5(3)e^{0.4(3)} + 31.25e^{0.4(3)}]$
$\quad – [2.5(1)^2e^{0.4(1)} – 12.5(1)e^{0.4(1)} + 31.25e^{0.4(1)}]$
$= [22.5e^{1.2} – 37.5e^{1.2} + 31.25e^{1.2}]$
$\quad – [2.5e^{0.4} – 12.5e^{0.4} + 31.25e^{0.4}]$
$= [16.25e^{1.2}] – [21.25e^{0.4}]$
≈ 22.250625

Ex. 4 $\int \sqrt{16x^2 - 9}\,dx$

Solution:

This integral seems to almost match integral #19. We first need to factor out a 16 from under the radical:

$$\int \sqrt{16x^2 - 9}\,dx = \int \sqrt{16(x^2 - \tfrac{9}{16})}\,dx = \int 4\sqrt{x^2 - \tfrac{9}{16}}\,dx$$

$$= 4\int \sqrt{x^2 - \tfrac{9}{16}}\,dx$$

Now, the integral matches #19:

$$\int \sqrt{x^2 \pm a^2}\,dx = \tfrac{1}{2}\left(x\sqrt{x^2 \pm a^2} \pm a^2 \ln\left| x + \sqrt{x^2 \pm a^2} \right|\right) + C$$

The value of $a = \frac{3}{4}$:

$$= 4\int \sqrt{x^2 - \tfrac{9}{16}}\,dx$$

$$= 4\bullet\tfrac{1}{2}\left(x\sqrt{x^2 - \left(\tfrac{3}{4}\right)^2} - \left(\tfrac{3}{4}\right)^2 \ln\left| x + \sqrt{x^2 - \left(\tfrac{3}{4}\right)^2} \right|\right) + C$$

$$= 2\left(x\sqrt{x^2 - \tfrac{9}{16}} - \tfrac{9}{16}\bullet\ln\left| x + \sqrt{x^2 - \tfrac{9}{16}} \right|\right) + C$$

Ex. 5 $\int \frac{dx}{\sqrt[3]{x}\,(9\sqrt[3]{x}+12)}$

Solution:

This integral seems to almost match integral #27. There is a common factor of 3 in the denominator, so let's first factor that out:

$$\int \frac{dx}{\sqrt[3]{x}\,(9\sqrt[3]{x}+12)} = \int \frac{dx}{3\sqrt[3]{x}\,(3\sqrt[3]{x}+4)}$$

Next, we need to make a substitution to get rid of the $\sqrt[3]{x}$.

Let $u = \sqrt[3]{x} = (x)^{1/3}$

$du = \frac{1}{3}(x)^{-2/3}\,dx = \frac{dx}{3\sqrt[3]{x^2}}$

The problem is that we only have a $\sqrt[3]{x}$ in the denominator and not a $\sqrt[3]{x^2}$. Let's try multiplying top and bottom of the integral by $\sqrt[3]{x}$:

$$\int \frac{dx}{3\sqrt[3]{x}\,(3\sqrt[3]{x}+4)} = \int \frac{dx}{3\sqrt[3]{x}\,(3\sqrt[3]{x}+4)}\,\frac{\sqrt[3]{x}}{\sqrt[3]{x}}$$

$$= \int \frac{\sqrt[3]{x}\;dx}{3\sqrt[3]{x^2}\,(3\sqrt[3]{x}+4)}$$

Now let's try our substitution:

Let $u = \sqrt[3]{x}$, then $du = \frac{dx}{3\sqrt[3]{x^2}}$

We will replace $\sqrt[3]{x}$ by u in the numerator and denominator:

$$\int \frac{\sqrt[3]{x}\;dx}{3\sqrt[3]{x^2}\,(3\sqrt[3]{x}+4)} = \int \frac{u\,dx}{3\sqrt[3]{x^2}\,(3u+4)}$$

Then we will replace $\frac{dx}{3\sqrt[3]{x^2}}$ by du:

$$\int \frac{u\,dx}{3\sqrt[3]{x^2}\,(3u+4)} = \int \frac{u\,du}{(3u+4)}$$

So, we are not going to be using integral # 27, but integral #25:

$$\int \frac{x}{ax+b}\,dx = \frac{x}{a} - \frac{b}{a^2}\ln\left| ax + b \right| + C$$

Our variable of integration is u instead of x, but otherwise it fits the pattern. The value of a = 3 and b = 4, so we get:

$$\int \frac{u\,du}{(3u+4)} = \frac{u}{3} - \frac{4}{(3)^2} \ln \left| 3u + 4 \right| + C$$

$$= \frac{u}{3} - \frac{4}{9} \ln \left| 3u + 4 \right| + C$$

Now, replace u by $\sqrt[3]{x}$ to get our final answer:

$$= \frac{u}{3} - \frac{4}{9} \ln \left| 3u + 4 \right| + C = \frac{\sqrt[3]{x}}{3} - \frac{4}{9} \ln \left| 3\sqrt[3]{x} + 4 \right| + C$$