Section 6.2 - Applications to Business & Economics

In this section and the next, we will be looking at several applications of the definite integral. If we think of the function we are integrating as rate function, then the definite integral would represent the accumulation of the total quantity. Let's explore some examples.

Ex. 1 A certain oil well that yields 100 barrels of crude oil a month will run dry in 2 years. The price of crude oil is \$120 per barrel and is expected to rise at a constant rate of 12 cents per barrel per month. If the oil is sold as soon as it is extracted from the ground, what will be the total future revenue from the well?

Solution:

The marginal revenue is rate in change in price times quantity sold. The price is \$120 + 0.12t where t is the number of months from now and the quantity sold is 100 barrels per month. Thus the revenue is increasing at a rate of (\$120 + 0.12t)•100. The well will run dry in two years which is twenty-four months. Thus, to find the total revenue, we will need to integrate the marginal revenue over the interval [0, 24]:

$$\int_0^{24} 100(120 + 0.12t)\,dt = \int_0^{24} (12000 + 12t)\,dt$$

$$= 12000t + 6t^2 \Big|_0^{24} = [12000(24) + 6(24)^2] - [0]$$

$$= \$291,456.$$

Hence, the total revenue from the well is \$291,456.

Ex. 2 The demand for a manufacturer's product is increasing exponentially at a rate of 2% per year. If the current demand is 5000 units per year and if the price remains fixed at \$400 per unit, how much revenue will the manufacturer receive from the sale of the product over the next two years?

Solution:

Since the demand is increasing exponentially at a rate of 2% per year, the change in demand function can be expressed as:

$d(t) = 5000e^{0.02t}$, where t is the number of years from now. The price is fixed \$400, so the marginal revenue will be equal to $400 \cdot 5000e^{0.02t}$ or $2000000e^{0.02t}$. Integrating, we find the total revenue to be:

$$\int_0^2 2000000e^{0.02t}\,dt = \left.\frac{2000000e^{0.02t}}{0.02}\right|_0^2 = \left.100000000e^{0.02t}\right|_0^2$$

$$= 100000000[e^{0.02(2)} - e^{0.02(0)}] = 100000000[e^{0.04} - 1]$$

$\approx 4{,}081{,}077.41$.

Thus, the total revenue is \$4,081,077.41.

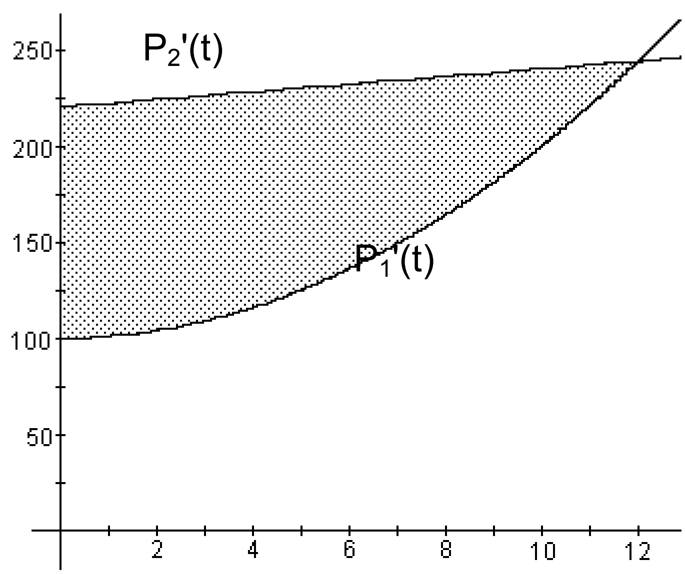
Ex. 3 Suppose that t years from now, one investment will generate a profit at a rate of $P_1'(t) = 100 + t^2$ hundred dollars per year, while a second investment will generate a profit at a rate of $P_2'(t) = 220 + 2t$ hundred dollars per year.

a) Sketch the rate of profitability curves $y = P_1'(t)$ and $y = P_2'(t)$ and shade the region whose area represent the net excess profit in investing in the second plan.

b) Compute the net excess profit in investing in the second plan until the second plan is no longer more profitable than the first plan.

Solution:

a) We begin by sketching the graph and determining where the two curves intersect:



Setting $P_1'(t) = P_2'(t)$ and solving yields:

$100 + t^2 = 220 + 2t$

$t^2 - 2t - 120 = 0$

$(t - 12)(t + 10) = 0$

$t = 12$ or $t = -10$.

But, $t = 12$ is the only solution that makes sense so the second investment is more profitable than the first one for the first twelve years.

b) To compute the excess profit, we will need to integrate $P_2'(t) - P_1'(t)$ from 0 to 12:

$$\int_0^{12} (220 + 2t) - (100 + t^2)\, dt = \int_0^{12} (-t^2 + 2t + 120)\, dt$$

$$= -\frac{t^3}{3} + t^2 + 120t \Big|_0^{12} = -\frac{(12)^3}{3} + (12)^2 + 120(12) - [0]$$

$= -576 + 144 + 1440 = 1008$ hundreds.

The excess profit is \$100,800.

There are some applications that involve finding the future value of an account with a continuous stream where the interest is compounded continuously. Recall that if interest is compounded continuously and the principal is fixed, the total amount in the account is found by using $Q(t) = Q_o e^{kt}$. Now, if money is actually flowing into an account continuously, we would then replace Q_o by the rate of money flow and integrate. Consider the following example:

Ex. 4 Money is transferred continuously into an account at a constant rate of \$1000 per year. The account earns interest at an annual rate of 8% compounded continuously. How much will be in the account after 5 years (i.e., what will be the future value)?

<u>Solution:</u>

The amount of money in the account is growing at a rate of $1000e^{0.08t}$ dollars per year where t is the time is years. To find the total amount in the account after 5 years, we will need to integrate :

$$\int_0^{5} 1000e^{0.08t}\, dt = \frac{1000}{0.08} e^{0.08t} \Big|_0^{5} = 12{,}500\, e^{0.08t} \Big|_0^{5}$$

$= 12{,}500(e^{0.08(5)} - e^{0.08(0)}) = 12{,}500(e^{0.4} - 1) \approx \$6147.81$

There will be about \$6,147.81 in the account after five years.

By the same token, it is useful to find out how much something is worth is terms of today's dollars. The present value of a fixed amount invested in account that pays interest compounded continuously can be found by using $Q = Q_o e^{-kt}$. For a continuous money flow problem, we would then replace Q_o by the rate of money flow and integrate. Let's find the present value of the account for the last example.

Ex. 5 Money is transferred continuously into an account at a constant rate of \$1000 per year. The account earns interest at an annual rate of 8% compounded continuously. If the money will be invested for five years, what is the present value of the account?

Solution:

The present value of the account is growing at a rate of $1000e^{-0.08t}$ dollars per year where t is the time is years. To find the present value of the account after 5 years, we will need to integrate:

$$\int_0^5 1000e^{-0.08t}\,dt = \frac{1000}{-0.08}e^{-0.08t}\Big|_0^5 = -12{,}500e^{-0.08t}\Big|_0^5$$

$$= -12{,}500(e^{-0.08(5)} - e^{-0.08(0)}) = -12{,}500(e^{-0.4} - 1)$$

$$\approx 4121.00$$

The present value of the account is about \$4121.00.

Another way to think of the present value is what fixed amount you would have to invest in an account with the same interest rate compounded continuously to obtain the same future value. In the last example, if we invested \$4121.00 in an account paying 8% interest compounded continuously, we would have: $4121.00e^{0.08(5)} = 4121e^{0.4} \approx \$6147.81$ in the account after five years. Notice that this answer matches the answer we had in example four.

Ex. 6 It is expected that t years from now, a 9-year franchise will be generating a profit at the rate of $f(t) = 10{,}000 + 500t$ dollars per year. If the prevailing annual interest rate remains fixed at 5% compounded continuously, what is the present value of the franchise?

Solution:

The present value of the franchise is growing at a rate of $(10{,}000 + 500t)e^{-0.05t}$ dollars per year where t is the time is years. To find the present value of the franchise after 9 years, we will need to integrate:

$$\int_0^9 (10{,}000 + 500t)e^{-0.05t}\,dt$$

$= \int_0^9 10{,}000e^{-0.05t}\,dt + \int_0^9 500te^{-0.05t}\,dt$

$= \frac{10000}{-0.05}e^{-0.05t}\Big|_0^9 + 500\int_0^9 te^{-0.05t}\,dt$

$= -200{,}000e^{-0.05t}\Big|_0^9 + \underline{500\int_0^9 te^{-0.05t}\,dt}$

To compute the second integral, we need to use integration by parts:

I) Let u = t and dv = $e^{-0.05t}$ dt.

II) Then du = dt and v = $\int e^{-0.05t}\,dt = \frac{e^{-0.05t}}{-0.05}$.

III) Plug into the integration by parts formula & integrate:

$\int u \bullet dv = u \bullet v - \int v \bullet du$

$500\int_0^9 te^{-0.05t}\,dt = 500t \bullet \frac{e^{-0.05t}}{-0.05} - 500\int_0^9 \frac{e^{-0.05t}}{-0.05}\,dt$

$= [-10000t\,e^{-0.05t}]\Big|_0^9 + 10000\int_0^9 e^{-0.05t}\,dt$

$= [-10000t\,e^{-0.05t}]\Big|_0^9 + 10000\frac{e^{-0.05t}}{-0.05}\Big|_0^9$

$= \underline{[-10000t\,e^{-0.05t} - 200000e^{-0.05t}]}\Big|_0^9$

Thus, the entire integral is:

$= [-200{,}000e^{-0.05t} - 10000t\,e^{-0.05t} - 200{,}000e^{-0.05t}]\Big|_0^9$

$= [-400{,}000e^{-0.05t} - 10000t\,e^{-0.05t}]\Big|_0^9$

$= [-400{,}000e^{-0.05(9)} - 10000(9)e^{-0.05(9)}]$
$\quad - [-400{,}000e^{-0.05(0)} - 10000(0)e^{-0.05(0)}]$

$= [-400{,}000e^{-0.45} - 90000e^{-0.45}] - [-400{,}000e^{0} - 0]$

$= [-490000e^{-0.45}] + 400000 \approx 87{,}562.21$

The present value of the franchise is \$87,562.21.

Ex. 7 The fraction of the people residing in a particular town t years from now is estimated to be f(t) = $e^{-0.04t}$. Currently, there are 20,000 people residing in the town and new people are arriving at a rate of 500 people per year. What will the town's population be ten years from now?

Solution:

Let t be the number of years from now. The population can be split into two parts. One part is the number of people currently in the town that stay ten years ($20000e^{-0.04(10)}$) and the second part is how many of the new people arriving t years from now will stay *10 – t* years. Since the newcomers is increasing at a rate of 500 people per year, the change in new people staying is $500e^{-0.04(10-t)}$. Thus, the total number of new people staying is

$\int_0^{10} 500e^{-0.04(10-t)}\,dt$. The total population is:

$$\text{Pop.} = 20000e^{-0.04(10)} + \int_0^{10} 500e^{-0.04(10-t)}\,dt$$

$$= 20000e^{-0.4} + \int_0^{10} 500e^{0.04t-0.4}\,dt$$

$$= 20000e^{-0.4} + 500\frac{e^{0.04t-0.4}}{0.04}\Big|_0^{10}$$

$$= 20000e^{-0.4} + 12500[e^{0.04(10)-0.4} - e^{0.04(0)-0.4}]$$

$$= 20000e^{-0.4} + 12500[e^{0} - e^{-0.4}]$$

$$= 20000e^{-0.4} + 12500 - 12500e^{-0.4}$$

$$= 7500e^{-0.4} + 12500 \approx 17{,}527$$

After ten years, the town will have ≈ 17,527 people.

Ex. 8 Calculate the rate (in cm^3 per second) at which blood flows through the artery of radius 0.1 cm if the speed of the blood r from the central axis is $8 - 800r^2$ cm/sec.

Solution:

For a circle of radius r, the speed of the blood will be the same for every point on the circumference of that circle. If you let dr represent the width of that circumference, then the area of that circumference is $2\pi r \cdot dr$. The rate of blood flow on that piece is $(8 - 800r^2) \cdot 2\pi r\,dr$. Integrating from 0 to 0.1 gives us the rate of blood flow:

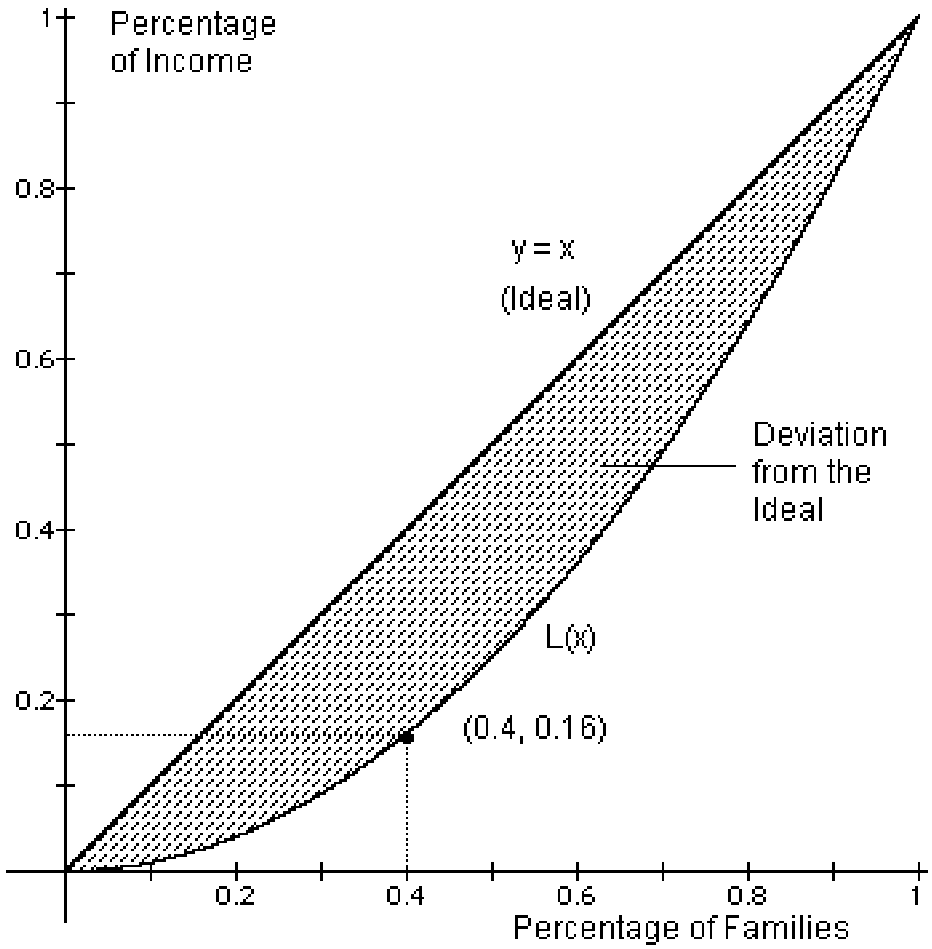
$$\int_0^{0.1} (8 - 800r^2)\cdot 2\pi r\,dr = 8\pi \int_0^{0.1} (2r - 200r^3)\,dr$$

$$= 8\pi[r^2 - 50r^4]\Big|_0^{0.1} = 8\pi([(0.1)^2 - 50(0.1)^4] - [0])$$

$$= 8\pi(0.01 - 0.005) = 0.04\pi \approx 0.0126$$

The rate of blood flow is about 0.0126 cm^3/s.

In economics, Lorentz Curves are used to measure the inequalities in the distribution of wealth. We typically graph the percentage of income on the vertical axis and the percentage of families on the horizontal axis. The ideal is the curve y = x since in this case, the income would be equally distributed. Here is an example of a typical Lorentz Curve L(x) compared to the ideal curve y = x.



If we consider the point (0.4, 0.16) on L(x), this means that the lowest 40% of all income receivers receive just 16% of the nation's wealth. The deviation from the ideal measures how far the wealth distribution is away from the ideal. A deviation of zero means that income distribution is perfectly equal and a deviation of 0.5 means that income distribution is perfectly unequal (one family has all the income). If you double the deviation, you get what is called the Gini Index:

Gini Index

If y = L(x) is the equation of a Lorentz curve, then the inequality in the corresponding distribution of income of wealth is measured by the Gini Index:

$$\text{Gini Index} = 2\int_0^1 [x - L(x)]\,dx$$

Ex. 9 Given the Lorentz Curve $L(x) = 0.55x^2 + 0.45x$, calculate the Gini Index.

Solution:

$$\text{Gini Index} = 2\int_0^1 [x - L(x)]\,dx$$

$$= 2\int_0^1 [x - (0.55x^2 + 0.45x)]dx = 2\int_0^1 [-0.55x^2 + 0.55x]dx$$

$$= 1.1\int_0^1 [-x^2 + x)]dx = 1.1[-\frac{x^3}{3} + \frac{x^2}{2}]\Big|_0^1$$

$$= 1.1[(-\frac{(1)^3}{3} + \frac{(1)^2}{2}) - (0)] = 1.1[\frac{1}{6}] = 0.14\overline{3}$$

Thus, the Gini Index is $0.14\overline{3}$.