Section 6.3 – Improper Integrals

Improper integrals are integrals where the limits of integration are not finite. Either the upper limit of integration is + ∞, the lower limit of integration is – ∞, or both. Since we can only integrate on a finite interval, we integrate as if we have a finite interval, replacing the infinite limit of integration with N, and take the limit as N → ∞ (or – ∞). If the limit is finite, we say that the integral converges. If the limit is not finite, we say the integral diverges. Let's summarize these ideas:

Improper Integrals
If f(x) is continuous, then

1) $\int_a^{+\infty} f(x)\,dx = \lim_{N\to+\infty} \int_a^N f(x)\,dx$

2) $\int_{-\infty}^{b} f(x)\,dx = \lim_{N\to-\infty} \int_N^b f(x)\,dx$

3) $\int_{-\infty}^{+\infty} f(x)\,dx = \lim_{N\to-\infty} \int_N^0 f(x)\,dx + \lim_{N\to+\infty} \int_0^M f(x)\,dx$

If the limit(s) are finite, the integral **converges**. If the limit(s) are infinite, the integral **diverges**.

Before we look at some examples, we will look at a property involving limits at infinity that occurs frequently in applications. We will not prove it, but assume it to be true:

Special Limit Property:
Let m > 0 and k > 0. Then $\lim_{N\to+\infty} N^m e^{-kN} = 0$.

Evaluate the following:

Ex. 1 $\int_1^{\infty} x^{-3/2}\,dx$

Solution:
First, we rewrite the integral as the limit as N → ∞ of the integral from 1 to N:

$$\int_1^{\infty} x^{-3/2}\,dx = \lim_{N\to+\infty} \int_1^N x^{-3/2}\,dx$$

Integrating, we get:

$$\lim_{N\to+\infty} \int_1^N x^{-3/2}\,dx = \lim_{N\to+\infty} \frac{x^{-\frac{1}{2}}}{-\frac{1}{2}} \Big|_1^N = \lim_{N\to+\infty} \frac{-2}{\sqrt{x}} \Big|_1^N$$

$$= \lim_{N\to+\infty} \frac{-2}{\sqrt{N}} - \frac{-2}{\sqrt{1}} = 0 + 2 = 2.$$ Hence, the integral converges.

Ex. 2 $\int_{-\infty}^{-2} \frac{2x}{x^2+5}\,dx$

Solution:

First, we rewrite the integral as the limit as $N \to -\infty$ of the integral from N to – 2:

$$\int_{-\infty}^{-2} \frac{2x}{x^2+5}\,dx = \lim_{N\to-\infty} \int_N^{-2} \frac{2x}{x^2+5}\,dx$$

Using substitution and integrating, we get:

Let $u = x^2 + 5$

$du = 2x\,dx$

Thus, $\lim_{N\to-\infty} \int_N^{-2} \frac{2x}{x^2+5}\,dx = \lim_{N\to-\infty} \int_N^{-2} \frac{1}{u}\,du$

$$= \lim_{N\to-\infty} \ln|u|\Big|_{x=N}^{x=-2} = \lim_{N\to-\infty} \ln|x^2+5|\Big|_N^{-2}$$

$$= \lim_{N\to-\infty} \ln|(-2)^2+5| - \ln|N^2+5|$$

$$= \lim_{N\to-\infty} \ln(9) - \ln(N^2+5) = -\infty.$$ Hence, the integral diverges.

Ex. 3 $\int_0^{\infty} xe^{-x}\,dx$

Solution:

First, we rewrite the integral as the limit as $N \to \infty$ of the integral from 0 to N:

$$\int_0^{\infty} xe^{-x}\,dx = \lim_{N\to+\infty} \int_0^N xe^{-x}\,dx$$

Using Integration by parts and integrating, we get:

I) Let $u = x$ and $dv = e^{-x}\,dx$.

II) Then du = dx and v = $\int e^{-x}\,dx = -e^{-x}$.

III) Plug into the integration by parts formula and integrate:

$$\int u \cdot dv = u \cdot v - \int v \cdot du$$

$$\lim_{N \to +\infty} \int_0^N xe^{-x}\,dx$$

$$= \lim_{N \to +\infty} \left[x \cdot (-e^{-x}) \Big|_0^N - \int_0^N -e^{-x} \cdot dx\right]$$

$$= \lim_{N \to +\infty} \left[-xe^{-x} \Big|_0^N + \int_0^N e^{-x}\,dx\right]$$

$$= \lim_{N \to +\infty} \left[-xe^{-x} - e^{-x} \Big|_0^N\right]$$

$$= \lim_{N \to +\infty} \left[-Ne^{-N} - e^{-N} - (-0 - e^{-0})\right]$$

$$= \lim_{N \to +\infty} \left[-Ne^{-N} - e^{-N} + 1\right]$$

But, by the Special Limit Property, $\lim_{N \to +\infty} -Ne^{-N} = 0$

So, we get:
$= -0 - 0 + 1 = 1$. The integral converges.

Ex. 4 Suppose an investment will generate \$3500 per year in perpetuity. If it is invested at 7% compounded continuously, what is the present value of the investment?

Solution:

This is a lot like the continuous income stream problems we worked in section 6.2. The only difference is that this investment goes forever. Thus, the present value is:

$$\int_0^\infty 3500e^{-0.07x}\,dx$$

We rewrite the integral as the limit as $N \to \infty$ of the integral from 0 to N:

$$\int_0^\infty 3500e^{-0.07x}\,dx = \lim_{N \to +\infty} \int_0^N 3500e^{-0.07x}\,dx$$

$$= \lim_{N \to +\infty} 3500 \int_0^N e^{-0.07x}\, dx$$

Integrating, we get:

$$= \lim_{N \to +\infty} 3500 \int_0^N e^{-0.07x}\, dx = \lim_{N \to +\infty} 3500 \frac{e^{-0.07x}}{-0.07} \Big|_0^N$$

$$= \lim_{N \to +\infty} -50000[e^{-0.07N} - e^{-0.07(0)}]$$

$$= \lim_{N \to +\infty} -50000[e^{-0.07N} - 1] = -50000[0 - 1]$$

= \$50,000.

The present value of the account is \$50,000.

Notice that our result is simply $\frac{Q_o}{k}$. This is not a coincidence since $\lim_{N \to +\infty} \int_0^N Q_o e^{-kx}\, dx = \frac{Q_o}{k}$.

Another important application in mathematics using improper integrals are probability density functions. If you are trying to determine the life span of a particular model of T.V., the life span X can be thought of as a continuous random variable since the life span will vary among individual T.V.'s. The probability density function P measure the probability that an individual T.V. will last a certain amount of time. For example, we may want to know how likely will a particular T.V. set last 3 to 5 years (denoted $P(3 \le X \le 5)$) or how likely will it last more than six years (denoted $P(X > 6)$). There are some basic properties that a function must satisfy in order to be a probability density function:

Probability Density Functions

Let X be a continuous random variable. Then f(x) is a probability density function if all the following are satisfied:

1) $f(x) \ge 0$ for all real x.
2) The total area under the graph of f(x) is 1.
3) The probability that X lies in [a, b] is

$$P(a \le X \le b) = \int_a^b f(x)\, dx$$

Let's look at an example.

Ex. 5 The life span of the CD House's radios is measured by a random variable X with probability density function $f(x) = 0.005e^{-0.005x}$, where x denotes the lifespan (in months) of a randomly selected radio.

a) What is the probability that the life span of a randomly selected radio will be between 48 months and 72 months?

b) What is the probability that the life span of a randomly selected radio will be greater than 84 months?

Solution:

a) $P(48 \le X \le 72) = \int_{48}^{72} 0.005e^{-0.005x}\,dx$

$= 0.005\frac{e^{-0.005x}}{-0.005}\Big|_{48}^{72} = -e^{-0.005x}\Big|_{48}^{72} =$

$= -e^{-0.005(72)} + e^{-0.005(48)} = -e^{-0.36} + e^{-0.24}$

$\approx 0.08895.$

Thus, there is 8.895% chance the radio will last between 48 and 72 months.

b) $P(X > 84) = \int_{84}^{\infty} 0.005e^{-0.005x}\,dx$

$= \lim_{N\to+\infty} \int_{84}^{N} 0.005e^{-0.005x}\,dx$

$= \lim_{N\to+\infty} 0.005\frac{e^{-0.005x}}{-0.005}\Big|_{84}^{N} = \lim_{N\to+\infty} -e^{-0.005x}\Big|_{84}^{N}$

$= \lim_{N\to+\infty} -e^{-0.005(N)} + e^{-0.005(84)}$

$= \lim_{N\to+\infty} -e^{-0.005(N)} + e^{-0.42} = 0 + e^{-0.42}$

$\approx 0.65705.$

Thus, there is 65.705% chance the radio will last longer than 84 months.