Sect 11.2 - Rational Exponents

Concept #1 Definition of $a^{1/n}$ and $a^{m/n}$.

In this section, we want to learn how to write radicals using exponential notation. To see how this is done, let's consider the $\sqrt{a}$ where a ≥ 0. We want to represent $\sqrt{a}$ as $a^{\text{some power}}$. Let p be that power. Thus, $\sqrt{a} = a^p$. Now, if we multiply the $\sqrt{a}$ by itself, we will get a:

$\sqrt{a} \bullet \sqrt{a} = a$ (substitute $\sqrt{a} = a^p$)

$a^p \bullet a^p = a$ (multiplying powers with the same base, add exponents)

$a^{p+p} = a$

$a^{2p} = a$ (the exponent of a is 1, thus 2p = 1)

$2p = 1$

$p = \frac{1}{2}$. Hence, $a^{1/2} = \sqrt{a}$

Now, let's try this same idea for $\sqrt[3]{b}$. Let $b^r = \sqrt[3]{b}$.

$\sqrt[3]{b} \bullet \sqrt[3]{b} \bullet \sqrt[3]{b} = b$ (substitute $\sqrt[3]{b} = b^r$)

$b^r \bullet b^r \bullet b^r = b$

$b^{r+r+r} = b$

$b^{3r} = b$

Hence, 3r = 1 which means $r = \frac{1}{3}$.

Notice a pattern. We are get the fraction 1/n for the exponent where n is equal to the index. Hence, in general, $\sqrt[n]{a} = a^{1/n}$ provided that the radical is defined.

Definition of $a^{1/n}$

Let n be a natural number greater than 1. If $\sqrt[n]{a}$ is a real number, then $a^{1/n} = \sqrt[n]{a}$.

Simplify the following:

Ex. 1a $(-64)^{1/3}$ Ex. 1b $81^{1/4}$

Ex. 1c $-144^{1/2}$ Ex. 1d $(-144)^{1/2}$

Ex. 1e $-(-32)^{-1/5}$

Solution:

a) $(-64)^{1/3} = \sqrt[3]{-64} = \sqrt[3]{(-4)^3} = -4$

b) $81^{1/4} = \sqrt[4]{81} = \sqrt[4]{3^4} = 3$

c) $-\,144^{1/2} = -\sqrt{144} = -\,12$

d) $(-\,144)^{1/2} = \sqrt{-144}$ which is undefined.

e) Recall that $a^{-r} = \frac{1}{a^r}$. So, $-\,(-\,32)^{-1/5} = -\,1\bullet(-\,32)^{-1/5} = \frac{-1}{(-32)^{1/5}}$

$= \frac{-1}{\sqrt[5]{-32}} = \frac{-1}{\sqrt[5]{(-2)^5}} = \frac{-1}{-2} = \frac{1}{2}$.

Recall that power rule for exponents: $(a^m)^n = a^{m\bullet n}$. If we are asked to simplify $(16^{1/4})^3$, we would first evaluate $16^{1/4}$ and then cube the result:

$(16^{1/4})^3 = (\sqrt[4]{16}\,)^3 = (2)^3 = 8$.

Similarly, if we had $(16^3)^{1/4}$, we would evaluate 16^3 and then take the fourth root of the result:

$(16^3)^{1/4} = \sqrt[4]{16^3} = \sqrt[4]{4096} = 8$.

Now, applying the power rule:

$(16^{1/4})^3 = 16^{(1/4)\bullet 3} = 16^{3/4}$ and $(16^3)^{1/4} = 16^{3(1/4)} = 16^{3/4}$

This means that $16^{3/4} = (\sqrt[4]{16}\,)^3 = \sqrt[4]{16^3} = 8$ from above. We can generalize this into a definition.

Definition of $a^{m/n}$ (rational exponents)

Let a be a real number and m and n be natural numbers where n > 1. If m/n is reduced to lowest terms and $\sqrt[n]{a}$ is a real number (defined), then

1) $a^{m/n} = (\sqrt[n]{a}\,)^m$ and

2) $a^{m/n} = \sqrt[n]{a^m}$

<u>Simplify the following:</u>

Ex. 2a $27^{2/3}$

Ex. 2b $25^{7/2}$

Ex. 2c $\left(\frac{1}{36}\right)^{5/2}$

Ex. 2d $(-\,2401)^{5/4}$

Ex. 2e $(-\,8)^{7/3}$

Solution:

a) Let's use part one of the definition:

$27^{2/3} = (\sqrt[3]{27}\,)^2 = (3)^2 = 9$

b) $25^{7/2} = (\sqrt{25}\,)^7 = (5)^7 = 78{,}125$

c) $\left(\frac{1}{36}\right)^{5/2} = \left(\sqrt{\frac{1}{36}}\right)^5 = \left(\frac{1}{6}\right)^5 = \frac{1}{7776}$.

d) $(-2401)^{5/4} = (\sqrt[4]{-2401})^5$, but $\sqrt[4]{-2401}$ is not a real number. Thus, $(-2401)^{5/4}$ is undefined in the real numbers.

e) $(-8)^{7/3} = (\sqrt[3]{-8})^7 = (-2)^7 = -128$.

Concept #2 Converting between Rational Exponents and Radical Notation

Convert the following into radical notation. Assume the variables represent positive real numbers:

Ex. 3a $a^{4/7}$ Ex. 3b $(7x^3)^{1/4}$

Ex. 3c $5a^{1/3}$ Ex. 3d $w^{-4/5}$

Solution:

a) $a^{4/7} = \sqrt[7]{a^4}$

b) $(7x^3)^{1/4} = \sqrt[4]{7x^3}$

c) $5a^{1/3} = 5\sqrt[3]{a}$

d) $w^{-4/5} = \frac{1}{w^{4/5}} = \frac{1}{\sqrt[5]{w^4}}$.

Convert the following into an expression using rational exponents. Assume the variables represent positive real numbers:

Ex. 4a $\sqrt[4]{b^5}$ Ex. 4b $6\sqrt[3]{5b}$

Ex. 4c $-3\sqrt[5]{a^2}$ Ex. 4d $(\sqrt[5]{-3a})^2$

Solution:

a) $\sqrt[4]{b^5} = b^{5/4}$

b) $6\sqrt[3]{5b} = 6(5b)^{1/3}$

c) $-3\sqrt[5]{a^2} = -3a^{2/5}$

d) $(\sqrt[5]{-3a})^2 = (-3a)^{2/5}$

Concept #3 Properties of Rational Exponents

The properties of exponents we developed in chapter 5 can be used with rational exponents.

Properties of Integral Exponents

Assume that a and b are non-zero real numbers and m and n are rational numbers such a^m, a^n, b^m, and b^n are real numbers.

Description	Property	Example
1. Multiplying like bases	The Product Rule 1. $a^m a^n = a^{m+n}$	$x^{2/5}x^{4/5} = x^{2/5+4/5} = x^{6/5}$
2. Dividing like bases	The Quotient Rule 2. $\frac{b^m}{b^n} = b^{m-n}$	$\frac{b^{6/7}}{b^{2/7}} = b^{6/7-2/7} = b^{4/7}$
3. A power raised by a power	The Power Rule 3. $(a^m)^n = a^{m \cdot n}$	$(x^{3/5})^{2/7} = x^{3/5 \cdot 2/7} = x^{6/35}$
4. A product to a power	Power of a Product Rule 4. $(ab)^n = a^n b^n$	$(ab)^{2/3} = a^{2/3}b^{2/3}$
5. A quotient to a power	Power of a Quotient Rule 5. $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$	$\left(\frac{a}{b}\right)^{4/5} = \frac{a^{4/5}}{b^{4/5}}$
Description	**Definition**	**Example**
1. Negative Exponents	1. $a^{-n} = \left(\frac{1}{a}\right)^n = \frac{1}{a^n}$	$x^{-7/3} = \frac{1}{x^{7/3}}$
2. Zero Exponents	2. $a^0 = 1$	$(-5)^0 = 1$

Use the properties of exponents to simplify the following:

Ex. 5a $\dfrac{x^{2/3} \bullet x^{1/4}}{x^{5/6}}$

Ex. 5b $\left(\dfrac{a^{2/3}b^{1/4}}{c^{3/8}}\right)^{12}$

Ex. 5c $\left(\dfrac{1875y^2z^{-7}}{3y^{-6}z^5}\right)^{1/4}$

Ex. 5d $\sqrt[3]{\sqrt{x}}$

Solution:

a) Multiplying powers of the same base, add the exponents:

$$\frac{x^{2/3}\bullet x^{1/4}}{x^{5/6}} = \frac{x^{2/3+1/4}}{x^{5/6}} = \frac{x^{11/12}}{x^{5/6}}$$

Dividing powers of the same base, subtract the exponents:

$$\frac{x^{11/12}}{x^{5/6}} = x^{11/12 - 5/6} = x^{1/12} \text{ or } \sqrt[12]{x}.$$

b) Apply the product and quotient to a power rules:

$$\left(\frac{a^{2/3}b^{1/4}}{c^{3/8}}\right)^{12} = \frac{(a^{2/3})^{12}(b^{1/4})^{12}}{(c^{3/8})^{12}} \qquad \text{(apply the power rule)}$$

$$= \frac{a^8b^3}{c^{9/2}}.$$

c) Simplify inside the parenthesis using the quotient rule:

$$\left(\frac{1875y^2z^{-7}}{3y^{-6}z^5}\right)^{1/4} = \left(\frac{625y^{2-(-6)}z^{-7-5}}{1}\right)^{1/4} = \left(\frac{625y^8z^{-12}}{1}\right)^{1/4}$$

Now, use the definition of negative exponents:

$$\left(\frac{625y^8z^{-12}}{1}\right)^{1/4} = \left(\frac{625y^8}{z^{12}}\right)^{1/4}$$

Apply the product and quotient to a power rule:

$$\left(\frac{625y^8}{z^{12}}\right)^{1/4} = \frac{(625)^{1/4}(y^8)^{1/4}}{(z^{12})^{1/4}}$$

Apply the power rule:

$$\frac{(625)^{1/4}(y^8)^{1/4}}{(z^{12})^{1/4}} = \frac{\sqrt[4]{625}\,(y^2)}{z^3} = \frac{5y^2}{z^3}.$$

d) $\sqrt[3]{\sqrt{x}} = (\sqrt{x})^{1/3} = (x^{1/2})^{1/3}$ (multiply the exponents)

$= x^{1/2\bullet 1/3} = x^{1/6} = \sqrt[6]{x}$.

Concept #4 Applications involving Rational Exponents

Solve the following:

Ex. 6 The radius r of a right circular cone of volume V and height h is given by $r = \left(\frac{3V}{\pi h}\right)^{1/2}$. Find the radius of a cone with a volume of 680.595 in^3 and a height of 9 in. Use $\pi \approx 3.14$.

Solution:

$$r = \left(\frac{3V}{\pi h}\right)^{1/2} = r = \left(\frac{3(680.595)}{(3.14)(9)}\right)^{1/2} = (72.25)^{1/2} = \sqrt{72.25} = 8.5$$

So, the radius is 8.5 inches.