# Sect 11.3 - Simplifying Radical Expressions

Concept #1 Multiplication and Division Properties of Radicals

We can use our properties of exponents to establish two properties of radicals:

$\sqrt[n]{ab} = (ab)^{1/n} = a^{1/n}b^{1/n} = \sqrt[n]{a} \bullet \sqrt[n]{b}$ & $\sqrt[n]{\frac{a}{b}} = \left(\frac{a}{b}\right)^{1/n} = \frac{a^{1/n}}{b^{1/n}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$

**Multiplication and Division Properties of Radicals**

Let a and b be real numbers such that $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are real numbers and $b \neq 0$. Then

1) Multiplication Property of Radicals: $\sqrt[n]{ab} = \sqrt[n]{a} \bullet \sqrt[n]{b}$

2) Division Property of Radicals: $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$

We can use these properties forwards and backwards to simplify radicals.

**Simplify the following:**

Ex. 1a $\sqrt{900}$ Ex. 1b $\sqrt[3]{\frac{27}{125}}$

Ex. 1c $\sqrt[4]{125} \bullet \sqrt[4]{5}$ Ex. 1d $\frac{\sqrt[5]{900000}}{\sqrt[5]{9}}$

Solution:

a) $\sqrt{900} = \sqrt{9 \bullet 100} = \sqrt{9} \bullet \sqrt{100} = 3 \bullet 10 = 30$

b) $\sqrt[3]{\frac{27}{125}} = \frac{\sqrt[3]{27}}{\sqrt[5]{125}} = \frac{3}{5}$

c) $\sqrt[4]{125} \bullet \sqrt[4]{5} = \sqrt[4]{125 \bullet 5} = \sqrt[4]{625} = 5$

d) $\frac{\sqrt[5]{900000}}{\sqrt[5]{9}} = \sqrt[5]{\frac{900000}{9}} = \sqrt[5]{100000} = 10$

In working with radical expressions where the radicand is not a perfect power of the index, it is important to simplify the radical as much as possible. With fractions, it is fairly simple to tell when a fraction is completely simplified. Let's think about the criteria for a radical to be completely simplified.

First, all factors of the radicand have powers that are less than the index. For example, $\sqrt{6}$ is simplified since $\sqrt{6} = \sqrt{2 \bullet 3}$ and the power of 2 and the power of 3 are 1 which is less than the index. But, $\sqrt{20}$ is not simplified since $\sqrt{20} = \sqrt{4 \bullet 5} = \sqrt{2^2 \bullet 5}$ and the power of 2 is equal to the index. In fact, $\sqrt{20} = \sqrt{4 \bullet 5} = \sqrt{4} \bullet \sqrt{5} = 2\sqrt{5}$. Second, the radicand has no fractions. For, instance, $\frac{\sqrt[3]{5}}{9}$ is simplified since the radicand $\sqrt[3]{5}$ has no fractions, whereas $\sqrt[3]{\frac{5}{9}}$ is not simplified since the radicand has a fraction. But, as we will see in a later section, we can rewrite $\sqrt[3]{\frac{5}{9}}$ as $\sqrt[3]{\frac{5}{9} \bullet 1} = \sqrt[3]{\frac{5}{9} \bullet \frac{3}{3}} = \sqrt[3]{\frac{15}{27}} = \frac{\sqrt[3]{15}}{\sqrt[3]{27}} = \frac{\sqrt[3]{15}}{3}$. Third, there can be no radicals in the denominator. An expression like $\frac{\sqrt{2}}{3}$ is simplified, but $\frac{2}{\sqrt{2}}$ is not since there is a radical in the denominator. To fix this problem, as we will see later, we can multiply top and bottom by $\sqrt{2}$: $\frac{2}{\sqrt{2}} = \frac{2}{\sqrt{2}} \bullet 1 = \frac{2}{\sqrt{2}} \bullet \frac{\sqrt{2}}{\sqrt{2}} = \frac{2\sqrt{2}}{\sqrt{4}} = \frac{2\sqrt{2}}{2} = \sqrt{2}$. Lastly, the exponents in the radicand and the index have 1 as the only common factor. A problem like $\sqrt[9]{x^2}$ is simplified since the only common factor of 2 and 9 is 1. But, $\sqrt[9]{x^3}$ is not simplified since 3 and 9 have a common factor of 3. In fact, $\sqrt[9]{x^3} = x^{3/9} = x^{1/3} = \sqrt[3]{x}$. Let's summarize these conditions.

**<u>Simplified Form of a Radical</u>**

A radical is simplified if all of the following conditions are met:

1) All factors of the radicand have powers that are less than the index.
2) The radicand has no fractions.
3) There can be no radicals in the denominator.
4) The exponents in the radicand and the index have 1 as the only common factor.

Concept #2 Simplifying Radicals Using the Multiplication Property of Radicals.

The key to simplify radicals is the separate out the factors that are perfect powers from the factors that are not perfect powers. We can use the multiplication property to do this and then simplify the perfect powers. For instance, in the $\sqrt[3]{40}$ , 40 has factors of 8 and 5. The factor of 8 is a perfect cube, so $\sqrt[3]{40} = \sqrt[3]{8 \bullet 5} = \sqrt[3]{8} \bullet \sqrt[3]{5} = \sqrt[3]{2^3} \bullet \sqrt[3]{5} = 2\sqrt[3]{5}$ . Also, if the index is less than the power of a factor of the radicand, we can divide that power by the index to figure out how to split the factors. The quotient raised to the power of the index will be the perfect power. The remainder will be the power of the factor that is not a perfect power. For instance, in $\sqrt{x^9}$ , to figure out how to split up $x^9$, divide 9 by 2:

$$\begin{array}{r|l} & \;\,4 \\ \hline 2 & \;\,9 \\ & -8 \\ \hline & \;\,1 \end{array}$$

The perfect power will be $(x^4)^2$ and the factor that is not a perfect power is $x^1$. Thus,

$$\sqrt{x^9} = \sqrt{(x^4)^2 \bullet x} = \sqrt{(x^4)^2} \bullet \sqrt{x} = x^4\sqrt{x}.$$

Let's try some additional examples:

**Simplify the following:**

Ex. 2a $\sqrt{25x^7y^3}$ Ex. 2b $\sqrt[3]{54a^4b^8}$

Ex. 2c $3\sqrt[4]{32x^4y^9z^{12}}$ Ex. 2d $\sqrt[3]{48x^9b^6}$

Solution:

a) Since 7 ÷ 2 = 3 R 1 and 3 ÷ 2 = 1 R 1, then

$\sqrt{25x^7y^3} = \sqrt{5^2(x^3)^2(y)^2xy}$ (apply the multiplication property)

$= \sqrt{5^2(x^3)^2(y)^2} \bullet \sqrt{xy}$ (simplify)

$= 5x^3y\sqrt{xy}$

b) Since 4 ÷ 3 = 1 R 1 and 8 ÷ 3 = 2 R 2 and 54 = 27•2 = $3^3$•2,

then $\sqrt[3]{54a^4b^8} = \sqrt[3]{3^3a^3(b^2)^3 \bullet 2ab^2}$ (apply the mult. property)

$= \sqrt[3]{3^3a^3(b^2)^3} \bullet \sqrt[3]{2ab^2}$ (simplify)

$= 3ab^2\sqrt[3]{2ab^2}$

c) Since 4 ÷ 4 = 1, 9 ÷ 4 = 2 R 1, 12 ÷ 4 = 3, and 32 = 16•2 = $2^4$•2, then

$3\sqrt[4]{32x^4y^9z^{12}} = 3\sqrt[4]{2^4x^4(y^2)^4(z^3)^4 \bullet 2y}$ (apply the mult. prop.)

$= 3\sqrt[4]{2^4x^4(y^2)^4(z^3)^4} \bullet \sqrt[4]{2y}$ (simplify)

$= 3|2xy^2z^3| \bullet \sqrt[4]{2y}$

But, since 2 and $y^2 \geq 0$, then they can come out of the absolute value:

$3|2xy^2z^3| \bullet \sqrt[4]{2y} = 3 \bullet 2y^2|xz^3|\sqrt[4]{2y} = 6y^2|xz^3|\sqrt[4]{2y}$.

d) Since 9 ÷ 3 = 3, 6 ÷ 3 = 2, and 48 = 16•3 = $2^4$•3 = $2^3$•2•3, then

$\sqrt[3]{48x^9b^6} = \sqrt[3]{2^3(x^3)^3(b^2)^3 \bullet 2 \bullet 3}$ (apply the mult. prop.)

$= \sqrt[3]{2^3(x^3)^3(b^2)^3} \bullet \sqrt[3]{2 \bullet 3}$ (simplify)

$= 2x^3b^2\sqrt[3]{6}$

Concept #3 Simplifying Radicals Using the Division Property of Radicals.

We will proceed in the same manner in simplifying radicals using the division property.

**Simplify the following; Assume the variables represent positive real numbers:**

Ex. 3a $\sqrt[3]{\dfrac{r^{11}}{r^5}}$ Ex. 3b $\dfrac{\sqrt[4]{5}}{\sqrt[4]{3125}}$ Ex. 3c $\dfrac{7\sqrt{72c^4d^2}}{12c}$

Ex. 3d $\dfrac{\sqrt[3]{375x^{14}y^7z^6}}{\sqrt[3]{3x^2y^{34}z^6}}$ Ex. 3e $\dfrac{\sqrt[3]{x} \bullet \sqrt[4]{x}}{\sqrt[5]{x}}$

Solution:

a) $\sqrt[3]{\dfrac{r^{11}}{r^5}}$ (simplify)

$= \sqrt[3]{r^6}$ (but 6 ÷ 3 = 2)

$= \sqrt[3]{(r^2)^3}$ (simplify)

$= r^2$

b) $\dfrac{\sqrt[4]{5}}{\sqrt[4]{3125}}$ (use the division property)

$= \sqrt[4]{\dfrac{5}{3125}}$ (reduce)

$= \sqrt[4]{\dfrac{1}{625}}$

$= \sqrt[4]{\dfrac{1}{5^4}} = \dfrac{1}{5}$.

c) $\dfrac{7\sqrt{72c^4d^2}}{12c} = \dfrac{7\sqrt{36(c^2)^2d^2 \bullet 2}}{12c}$ (use the multiplication property)

$= \dfrac{7\sqrt{36(c^2)^2d^2} \bullet \sqrt{2}}{12c}$ (simplify)

$= \dfrac{7 \bullet 6\left|c^2d\right| \bullet \sqrt{2}}{12c}$ (But c and d are > 0 from the directions)

$= \dfrac{42c^2d\sqrt{2}}{12c}$ (reduce)

$= \dfrac{7cd\sqrt{2}}{2}$.

d) $\dfrac{\sqrt[3]{375x^{14}y^7z^6}}{\sqrt[3]{3x^2y^{34}z^6}}$ (use the quotient rule)

$= \sqrt[3]{\dfrac{375x^{14}y^7z^6}{3x^2y^{34}z^6}}$ (reduce)

$= \sqrt[3]{\dfrac{125x^{12}}{y^{27}}}$

$= \sqrt[3]{\dfrac{5^3(x^4)^3}{(y^9)^3}}$ (simplify)

$= \dfrac{5x^4}{y^9}$

e) The indices are not the same so we cannot use our properties. But,

$\dfrac{\sqrt[3]{x} \bullet \sqrt[4]{x}}{\sqrt[5]{x}} = \dfrac{x^{1/3} \bullet x^{1/4}}{x^{1/5}} = x^{1/3 + 1/4 - 1/5}$

$= x^{20/60 + 15/60 - 12/60} = x^{23/60} = \sqrt[60]{x^{23}}$ .