Sect 11.5 - Multiplication of Radicals

Concept #1 Multiplication Property of Radicals.

Recall the Multiplication Property of Radicals introduced in Sect 11.3:

Multiplication Property of Radicals

Let a and b be real numbers such that $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are real numbers. Then

1) Multiplication Property of Radicals: $\sqrt[n]{ab} = \sqrt[n]{a} \bullet \sqrt[n]{b}$

We will be making extensive use of this property along with the commutative and associative properties of multiplication to multiply radical expressions.

Simplify the following. Assume the variables represent positive real numbers:

Ex. 1a $(6\sqrt{15})(-3\sqrt{10})$ Ex. 1b $(-\frac{7}{9}x\sqrt{y})(-3\sqrt{xy})$

Ex. 1c $(-11\sqrt[3]{9g^2h})(5\sqrt[3]{6gh})$ Ex. 1c $\sqrt{6} \bullet \sqrt{21} - 4\sqrt{56}$

Solution:

a) $(6\sqrt{15})(-3\sqrt{10})$ (use the commutative & associative prop.)

$= (6\bullet(-3))(\sqrt{15} \bullet \sqrt{10})$ (simplify, mult. prop.)

$= -18\sqrt{150}$

$= -18\sqrt{25 \bullet 6} = -18\sqrt{5^2 \bullet 6}$

$= -18\bullet 5\sqrt{6}$

$= -90\sqrt{6}$

b) $(-\frac{7}{9}x\sqrt{y})(-3\sqrt{xy})$ (use the comm. & assoc. prop.)

$= (-\frac{7}{9}x)\bullet(-3)(\sqrt{y} \bullet \sqrt{xy})$ (simplify, mult. prop.)

$= \frac{7}{3}x\sqrt{xy^2}$

$= \frac{7}{3}x|y|\sqrt{x}$ But y > 0 by the directions, so $|y| = y$:

$= \frac{7}{3}xy\sqrt{x}$

c) $(-11\sqrt[3]{9g^2h})(5\sqrt[3]{6gh})$ (use the comm. & assoc. prop.)

$= (-11\bullet 5)(\sqrt[3]{9g^2h}\bullet\sqrt[3]{6gh})$ (simplify, mult. prop.)

$= -55\sqrt[3]{54g^3h^2}$

$= -55\sqrt[3]{27g^3\bullet 2h^2} = -55\sqrt[3]{3^3g^3\bullet 2h^2}$

$= -55\bullet 3g\sqrt[3]{2h^2}$

$= -165g\sqrt[3]{2h^2}$.

d) The order of operations says to multiply before adding:

$\sqrt{6}\bullet\sqrt{21} - 4\sqrt{56}$ (mult. prop.)

$= \sqrt{126} - 4\sqrt{56}$

$= \sqrt{9\bullet 14} - 4\sqrt{4\bullet 14} = \sqrt{3^2\bullet 14} - 4\sqrt{2^2\bullet 14}$

$= 3\sqrt{14} - 4\bullet 2\sqrt{14}$

$= 3\sqrt{14} - 8\sqrt{14} = -5\sqrt{14}$

Keep in mind multiplying radicals is very different from adding and subtracting radicals like we did in the last section. Consider the following:

Simplify the following:

Ex. 2a $\sqrt{5}\bullet\sqrt{5}$

Ex. 2b $\sqrt{5} + \sqrt{5}$

Ex. 2c $\sqrt{6} + \sqrt{10}$

Ex. 2d $\sqrt{6}\bullet\sqrt{10}$

Solution:

a) Using the multiplication property, we get:

$\sqrt{5}\bullet\sqrt{5} = \sqrt{5^2} = 5$

b) We are adding two terms that are like radicals, so

$\sqrt{5} + \sqrt{5} = 1\sqrt{5} + 1\sqrt{5} = 2\sqrt{5}$

c) These two terms are not like radicals, so we cannot simplify:

$\sqrt{6} + \sqrt{10} = \sqrt{6} + \sqrt{10}$

d) Using the multiplication property, we get:

$\sqrt{6}\bullet\sqrt{10} = \sqrt{60} = \sqrt{4\bullet 15} = \sqrt{2^2\bullet 15} = 2\sqrt{15}$

If we are multiplying radical expressions with more than one term, we will need to use the distributive property in additional to the other properties.

<u>Simplify the following:</u>

Ex. 3a $6\sqrt{2}(3 - \sqrt{2})$ Ex. 3b $(\sqrt{7} + 4\sqrt{3})(2\sqrt{7} - \sqrt{3})$

Ex. 3c $(4\sqrt{5} - 3\sqrt{15})(6 - 7\sqrt{5} + 2\sqrt{15})$

Ex. 3d $(x\sqrt{x} - y\sqrt{y})(y\sqrt{x} + x\sqrt{y})$ Assume x and y are positive.

Ex. 3e $(\sqrt[3]{r} + \sqrt[3]{s})(\sqrt[3]{r^2} - \sqrt[3]{rs} + \sqrt[3]{s^2})$

Solution:

a) $6\sqrt{2}\ (3 - \sqrt{2})$ (distribute the $6\sqrt{2}$)

$= 6\sqrt{2}\ (3) - 6\sqrt{2}\ (\sqrt{2})$ (simplify)

$= 18\sqrt{2} - 6\sqrt{2^2}$

$= 18\sqrt{2} - 6\bullet 2$

$= 18\sqrt{2} - 12$

b) $(\sqrt{7} + 4\sqrt{3})(2\sqrt{7} - \sqrt{3})$ (FOIL)

$= \sqrt{7}\bullet 2\sqrt{7} - \sqrt{7}\bullet\sqrt{3} + 4\sqrt{3}\bullet 2\sqrt{7} - 4\sqrt{3}\bullet\sqrt{3}$ (simplify)

$= 2\sqrt{7^2} - \sqrt{21} + 8\sqrt{21} - 4\sqrt{3^2}$

$= 2\bullet 7 - \sqrt{21} + 8\sqrt{21} - 4\bullet 3$

$= 14 - \sqrt{21} + 8\sqrt{21} - 12$

$= 2 + 7\sqrt{21}$

c) $(4\sqrt{5} - 3\sqrt{15})(6 - 7\sqrt{5} + 2\sqrt{15})$ (distribute)

$= 4\sqrt{5}(6 - 7\sqrt{5} + 2\sqrt{15}) - 3\sqrt{15}\ (6 - 7\sqrt{5} + 2\sqrt{15})$

$= 4\sqrt{5}\ (6) - 4\sqrt{5}\bullet 7\sqrt{5} + 4\sqrt{5}\bullet 2\sqrt{15}$

$- 3\sqrt{15}\ (6) + 3\sqrt{15}\bullet 7\sqrt{5} - 3\sqrt{15}\bullet 2\sqrt{15}$

$= 24\sqrt{5} - 28\sqrt{5^2} + 8\underline{\sqrt{75}} - 18\sqrt{15} + 21\underline{\sqrt{75}} - 6\sqrt{15^2}$

But, $\sqrt{75} = \sqrt{25\bullet 3} = \sqrt{5^2\bullet 3} = 5\sqrt{3}$, so

$= 24\sqrt{5} - 28\sqrt{5^2} + 8\bullet\underline{5\sqrt{3}} - 18\sqrt{15} + 21\bullet\underline{5\sqrt{3}} - 6\sqrt{15^2}$

$= 24\sqrt{5} - 28\bullet 5 + 8\bullet 5\sqrt{3} - 18\sqrt{15} + 21\bullet 5\sqrt{3} - 6\bullet 15$

$= 24\sqrt{5} - 140 + 40\sqrt{3} - 18\sqrt{15} + 105\sqrt{3} - 90$

$= -230 + 145\sqrt{3} + 24\sqrt{5} - 18\sqrt{15}$.

d) $(x\sqrt{x} - y\sqrt{y})(y\sqrt{x} + x\sqrt{y})$ (FOIL & simplify)
$= x\sqrt{x} \bullet y\sqrt{x} + x\sqrt{x} \bullet x\sqrt{y} - y\sqrt{y} \bullet y\sqrt{x} - y\sqrt{y} \bullet x\sqrt{y}$
$= xy\sqrt{x^2} + x^2\sqrt{xy} - y^2\sqrt{xy} - xy\sqrt{y^2}$
Since x and y are positive, then $\sqrt{x^2} = x$ and $\sqrt{y^2} = y$:
$= xy \bullet x + x^2\sqrt{xy} - y^2\sqrt{xy} - xy \bullet y$
$= x^2y - xy^2 + x^2\sqrt{xy} - y^2\sqrt{xy}$
We cannot simplify since there are no terms that are both like terms and like radicals. Thus, our answer is
$x^2y - xy^2 + x^2\sqrt{xy} - y^2\sqrt{xy}$.

e) $(\sqrt[3]{r} + \sqrt[3]{s})(\sqrt[3]{r^2} - \sqrt[3]{rs} + \sqrt[3]{s^2})$ (distribute)
$= (\sqrt[3]{r})(\sqrt[3]{r^2} - \sqrt[3]{rs} + \sqrt[3]{s^2}) + (\sqrt[3]{s})(\sqrt[3]{r^2} - \sqrt[3]{rs} + \sqrt[3]{s^2})$
$= (\sqrt[3]{r})(\sqrt[3]{r^2}) - (\sqrt[3]{r})(\sqrt[3]{rs}) + (\sqrt[3]{r})(\sqrt[3]{s^2})$
$+ (\sqrt[3]{s})(\sqrt[3]{r^2}) - (\sqrt[3]{s})(\sqrt[3]{rs}) + (\sqrt[3]{s})(\sqrt[3]{s^2})$
$= \sqrt[3]{r^3} - \sqrt[3]{r^2s} + \sqrt[3]{rs^2} + \sqrt[3]{r^2s} - \sqrt[3]{rs^2} + \sqrt[3]{s^3}$
$= r - \sqrt[3]{r^2s} + \sqrt[3]{rs^2} + \sqrt[3]{r^2s} - \sqrt[3]{rs^2} + s$
$= r + s$

Notice #3e look like the sum of cubes: $F^3 + L^3 = (F + L)(F^2 - FL + L^2)$.

Concept #2 Expressions of the Form $(\sqrt[n]{a})^n$.

In section 11.1, we saw that $\sqrt[n]{a^n}$ was equal a if n was odd and $|a|$ if n was even. Similarly, if $\sqrt[n]{a}$ is a real number, then
$$(\sqrt[n]{a})^n = (a^{1/n})^n = a^{n/n} = a^1 = a$$

Simplify the following. Assume the variables represent positive real numbers:

Ex. 4a $(\sqrt{11})^2$ Ex. 4b $(\sqrt[7]{rz})^7$

Ex. 4c $(\sqrt[3]{x+y})^3$

Solution:

a) $(\sqrt{11})^2 = 11$

b) $(\sqrt[7]{rz})^7 = rz$

c) $(\sqrt[3]{x+y})^3 = x + y$

Concept #3 Special Case Products

Recall the special products:

1) $(F + L)^2 = F^2 + 2FL + L^2$
2) $(F - L)^2 = F^2 - 2FL + L^2$
3) $(F - L)(F + L) = F^2 - L^2$ {(F + L) and (F – L) are called conjugates}
4) $(F + L)(F^2 - FL + L^2) = F^3 + L^3$
5) $(F - L)(F^2 + FL + L^2) = F^3 - L^3$

We can use these special products to simplify expressions involving radicals if the expression fits one of these patterns:

Simplify the following. Assume the variables represent positive real numbers:

Ex. 5a $(\sqrt[3]{x} - \sqrt[3]{2y})(\sqrt[3]{x^2} + \sqrt[3]{2xy} + \sqrt[3]{4y^2})$

Ex. 5b $(3\sqrt{3x} - 2\sqrt{5})^2$

Ex. 5c $(\frac{2}{3}\sqrt{x} - \frac{4}{9}\sqrt{5w})(\frac{2}{3}\sqrt{x} + \frac{4}{9}\sqrt{5w})$

Ex. 5d $(x\sqrt{x} + y\sqrt{y})^2$

Solution:

a) $(\sqrt[3]{x} - \sqrt[3]{2y})(\sqrt[3]{x^2} + \sqrt[3]{2xy} + \sqrt[3]{4y^2})$ is a difference of cubes (#5), where $F = \sqrt[3]{x}$ and $L = \sqrt[3]{2y}$. Using our pattern, we get: $F^3 - L^3 = (\sqrt[3]{x})^3 - (\sqrt[3]{2y})^3$
$= x - 2y$.

b) $(3\sqrt{3x} - 2\sqrt{5})^2$ is a perfect square trinomial (#2), where $F = 3\sqrt{3x}$ and $L = 2\sqrt{5}$. Using our pattern, we get:

$= F^2 - 2FL + L^2 = (3\sqrt{3x})^2 - 2(3\sqrt{3x})(2\sqrt{5}) + (2\sqrt{5})^2$
$= 9(\sqrt{3x})^2 - 2(3\sqrt{3x})(2\sqrt{5}) + 4(\sqrt{5})^2$
$= 9(3x) - 2(3\sqrt{3x})(2\sqrt{5}) + 4(5)$
$= 27x - 12\sqrt{15x} + 20$

c) $\left(\frac{2}{3}\sqrt{x} - \frac{4}{9}\sqrt{5w}\right)\left(\frac{2}{3}\sqrt{x} + \frac{4}{9}\sqrt{5w}\right)$ is a difference of squares (#3) with F = $\frac{2}{3}\sqrt{x}$ and L = $\frac{4}{9}\sqrt{5w}$). Using our pattern, we get: $F^2 - L^2 = \left(\frac{2}{3}\sqrt{x}\right)^2 - \left(\frac{4}{9}\sqrt{5w}\right)^2$

$= \frac{4}{9}(\sqrt{x})^2 - \frac{16}{81}(\sqrt{5w})^2$

$= \frac{4}{9}x - \frac{16}{81}(5w)$

$= \frac{4}{9}x - \frac{80}{81}w$

d) $(x\sqrt{x} + y\sqrt{y})^2$ is a perfect square trinomial (#1), where F = $x\sqrt{x}$ and L = $y\sqrt{y}$. Using our pattern, we get:

$= F^2 + 2FL + L^2 = (x\sqrt{x})^2 + 2(x\sqrt{x})(y\sqrt{y}) + (y\sqrt{y})^2$

$= x^2(\sqrt{x})^2 + 2(x\sqrt{x})(y\sqrt{y}) + y^2(\sqrt{y})^2$

$= x^2(x) + 2(x\sqrt{x})(y\sqrt{y}) + y^2(y)$

$= x^3 + 2xy\sqrt{xy} + y^3$

Note that we could have worked example #5 without using the special products. We would proceed just like our calculations in example #3. For instance, $(x\sqrt{x} + y\sqrt{y})^2 = (x\sqrt{x} + y\sqrt{y})(x\sqrt{x} + y\sqrt{y})$ (FOIL)

$= x\sqrt{x} \bullet x\sqrt{x} + x\sqrt{x} \bullet y\sqrt{y} + y\sqrt{y} \bullet x\sqrt{x} + y\sqrt{y} \bullet y\sqrt{y}$

$= x^2\sqrt{x^2} + xy\sqrt{xy} + xy\sqrt{xy} + y^2\sqrt{y^2}$

$= x^2(x) + xy\sqrt{xy} + xy\sqrt{xy} + y^2(y)$

$= x^3 + xy\sqrt{xy} + xy\sqrt{xy} + y^3$

$= x^3 + 2xy\sqrt{xy} + y^3$

Concept #4 Multiply Radicals with Different Indices.

The multiplication and division property of radicals only works if the indices are the same. If the indices are different, we will need to convert the radicals into expressions with rational exponents and then use our properties of exponents to simplify.

Simplify the following:

Ex. 6a $\sqrt[3]{7} \bullet \sqrt[4]{7}$ Ex. 6b $\sqrt[3]{4} \bullet \sqrt[5]{9}$

Ex. 6c $\dfrac{\sqrt[6]{3x^2+2}}{\sqrt[8]{3x^2+2}}$

Solution:

a) $\sqrt[3]{7} \bullet \sqrt[4]{7} = 7^{1/3} \bullet 7^{1/4} = 7^{1/3 + 1/4} = 7^{7/12} = \sqrt[12]{7^7} = \sqrt[12]{823{,}543}$.

b) $\sqrt[3]{4} \bullet \sqrt[5]{9} = 4^{1/3} \bullet 9^{1/5}$. Neither bases nor the indices are the same, so we will make the indices the same by getting a common denominator:

$$4^{1/3} \bullet 9^{1/5} = 4^{5/15} \bullet 9^{3/15} = 4^{5 \cdot 1/15} \bullet 9^{3 \cdot 1/15} = (4^5 \bullet 9^3)^{1/15} = \sqrt[15]{4^5 \bullet 9^3}$$

c) $\dfrac{\sqrt[6]{3x^2+2}}{\sqrt[8]{3x^2+2}} = \dfrac{(3x^2+2)^{1/6}}{(3x^2+2)^{1/8}} = (3x^2 + 2)^{1/6 - 1/8} = (3x^2 + 2)^{4/24 - 3/24}$

$= (3x^2 + 2)^{1/24} = \sqrt[24]{3x^2+2}$.