Sect 11.6 - Rationalization

Concept #1 Simplified Form of a Radical.

Recall that in section 11.3, we stated the following criteria for determining is a radical is completely simplified:

Simplified Form of a Radical

A radical is simplified if all of the following conditions are met:

1) All factors of the radicand have powers that are less than the index.
2) The radicand has no fractions.
3) There can be no radicals in the denominator.
4) The exponents in the radicand and the index have 1 as the only common factor.

Up to this point, we have not addressed what we need to do if there are radicals in the denominator. Part of that will depend on if we have one term in the denominator or more than one term. The process involved with get rid of the radical in the denominator is called **rationalizing the denominator**. This is done by multiply the top and bottom by a form of 1. The word “rationalizing” simply means to make the number rational.

Concept #2 Rationalizing the Denominator that has One Term

If the denominator has one term, to get rid of the radical, we will need to multiply the radical by a quantity that will make the radicand in the denominator a perfect power. Whatever we multiply the denominator by, we will also need to multiply the numerator by the same quantity.

Multiply each radical by a factor that will rationalize it:

Ex. 1a $\sqrt{5} \bullet ? = 5$ Ex. 1b $\sqrt[3]{25h} \bullet ? = 5h$

Ex. 1c $\sqrt[4]{3x^3y} \bullet ? = 3xy$ where x and y are positive

Ex. 1d $\sqrt[5]{27a^2b^4c} \bullet ? = 3abc$

Solution:

a) To make 5 a perfect square, we need to multiply by another factor of 5: $\sqrt{5} \bullet \sqrt{5} = \sqrt{5^2} = 5$.

So, the missing factor is $\sqrt{5}$.

b) Since $25h = 5^2h$, to make this a perfect cube, we need to multiply by a factor of $5h^2$:

$$\sqrt[3]{5^2h} \bullet \sqrt[3]{5h^2} = \sqrt[3]{25h} \bullet \sqrt[3]{5h^2} = \sqrt[3]{5^3h^3} = 5h$$

So, missing factor is $\sqrt[3]{5h^2}$.

c) To make $3x^3y$ a perfect fourth power, we need to multiply it by a factor of 3^3xy^3:

$$\sqrt[4]{3x^3y} \bullet \sqrt[4]{3^3xy^3} = \sqrt[4]{3^4x^4y^4} = 3xy$$

So, missing factor is $\sqrt[4]{3^3xy^3}$.

d) Since $27a^2b^4c = 3^3a^2b^4c$, to make this a perfect fifth power, we need to multiply by a factor of $3^2a^3bc^4$:

$$\sqrt[5]{27a^2b^4c} \bullet \sqrt[5]{3^2a^3bc^4} = \sqrt[5]{3^3a^2b^4c} \bullet \sqrt[5]{3^2a^3bc^4}$$

$$= \sqrt[5]{3^5a^5b^5c^5} = 3abc$$

So, the missing factor is $\sqrt[5]{3^2a^3bc^4}$.

Rationalize the denominator:

Ex. 2a $\frac{10}{\sqrt{5}}$

Ex. 2b $\sqrt[3]{\frac{2r}{25h}}$

Ex. 2c $\frac{21}{\sqrt[4]{3x^3y}}$

Ex. 2d $\frac{\sqrt[5]{2c^3}}{\sqrt[5]{27a^2b^4c}}$

Ex. 2e $\sqrt{\frac{27}{11}}$

Ex. 2e $\frac{\sqrt[5]{729x^7}}{\sqrt[5]{3x^2}}$

Solution:

We can use our results from example #1 to help us for parts a - d.

a) $\frac{10}{\sqrt{5}}$ (rationalize the denominator)

$= \frac{10}{\sqrt{5}} \bullet \frac{\sqrt{5}}{\sqrt{5}}$ (simplify)

$= \frac{10\sqrt{5}}{\sqrt{5^2}} = \frac{10\sqrt{5}}{5} = 2\sqrt{5}$.

b) $\sqrt[3]{\frac{2r}{25h}} = \frac{\sqrt[3]{2r}}{\sqrt[3]{5^2h}}$ (rationalize the denominator)

$= \frac{\sqrt[3]{2r}}{\sqrt[3]{5^2h}} \bullet \frac{\sqrt[3]{5h^2}}{\sqrt[3]{5h^2}}$ (simplify)

$= \frac{\sqrt[3]{10rh^2}}{\sqrt[3]{5^3h^3}} = \frac{\sqrt[3]{10rh^2}}{5h}$.

c) $\frac{21}{\sqrt[4]{3x^3y}}$ (rationalize the denominator)

$= \frac{21}{\sqrt[4]{3x^3y}} \bullet \frac{\sqrt[4]{3^3xy^3}}{\sqrt[4]{3^3xy^3}}$ (simplify)

$= \frac{21\sqrt[4]{3^3xy^3}}{\sqrt[4]{3^4x^4y^4}} = \frac{21\sqrt[4]{3^3xy^3}}{3xy} = \frac{7\sqrt[4]{27xy^3}}{xy}$.

d) $\frac{\sqrt[5]{2c^3}}{\sqrt[5]{27a^2b^4c}} = \frac{\sqrt[5]{2c^3}}{\sqrt[5]{3^3a^2b^4c}}$ (rationalize the denominator)

$= \frac{\sqrt[5]{2c^3}}{\sqrt[5]{3^3a^2b^4c}} \bullet \frac{\sqrt[5]{3^2a^3bc^4}}{\sqrt[5]{3^2a^3bc^4}}$ (simplify)

$= \frac{\sqrt[5]{2\bullet 3^2a^3bc^7}}{\sqrt[5]{3^5a^5b^5c^5}} = \frac{\sqrt[5]{18a^3bc^7}}{3abc}$ But, $c^7 = c^5 \bullet c^2$, so

$\frac{\sqrt[5]{18a^3bc^7}}{3abc} = \frac{\sqrt[5]{18a^3bc^5 \bullet c^2}}{3abc} = \frac{c\sqrt[5]{18a^3bc^2}}{3abc} = \frac{\sqrt[5]{18a^3bc^2}}{3ab}$.

e) $\sqrt{\frac{27}{11}}$ (simplify)

$= \frac{\sqrt{27}}{\sqrt{11}} = \frac{\sqrt{3^2\bullet 3}}{\sqrt{11}} = \frac{3\sqrt{3}}{\sqrt{11}}$ (rationalize the denominator)

$= \frac{3\sqrt{3}}{\sqrt{11}} \bullet \frac{\sqrt{11}}{\sqrt{11}}$ (simplify)

$= \frac{3\sqrt{33}}{\sqrt{11^2}} = \frac{3\sqrt{33}}{11}$.

f) Notice that the numerator and denominator have common factors:

$$\frac{\sqrt[5]{729x^7}}{\sqrt[5]{3x^2}} = \sqrt[5]{\frac{729x^7}{3x^2}} = \sqrt[5]{243x^5} = \sqrt[5]{3^5x^5} = 3x.$$

Concept #3 Rationalizing the Denominator that has Two Terms.

Recall the formula for the difference of squares: $(F - L)(F + L) = F^2 - L^2$ where $(F - L)$ and $(F + L)$ are conjugates. There was no middle term in the answer since the two middle terms generated by the product were opposites. When we applied this with radicals in section 11.5 (example #5c), we did not get any radicals in the answer:

$(\frac{2}{3}\sqrt{x} - \frac{4}{9}\sqrt{5w})(\frac{2}{3}\sqrt{x} + \frac{4}{9}\sqrt{5w})$ is a difference of squares (#3) with $F = \frac{2}{3}\sqrt{x}$ and $L = \frac{4}{9}\sqrt{5w}$). Using our pattern, we get:

$$F^2 - L^2 = \left(\frac{2}{3}\sqrt{x}\right)^2 - \left(\frac{4}{9}\sqrt{5w}\right)^2$$
$$= \frac{4}{9}(\sqrt{x})^2 - \frac{16}{81}(\sqrt{5w})^2$$
$$= \frac{4}{9}x - \frac{16}{81}(5w)$$
$$= \frac{4}{9}x - \frac{80}{81}w.$$

So, if we want to rationalize the denominator that has two terms, we need to multiply it by its conjugate. Whatever we multiply the denominator by, we will also need to multiply the numerator by the same quantity.

<u>Multiply each expression by an expression that will rationalize it:</u>

Ex. 3a $(5 + \sqrt{7})\bullet ? = 18$ Ex. 3b $(\sqrt{2} - 5)\bullet ? = -23$

Solution:

a) The conjugate of $5 + \sqrt{7}$ is $5 - \sqrt{7}$, so
$(5 + \sqrt{7})(5 - \sqrt{7}) = (5)^2 - (\sqrt{7})^2 = 25 - 7 = 18$
So, the expression is $(5 - \sqrt{7})$.

b) The conjugate of $\sqrt{2} - 5$ is $\sqrt{2} + 5$, so
$(\sqrt{2} - 5)(\sqrt{2} + 5) = (\sqrt{2})^2 - (5)^2 = 2 - 25 = -23$
So, the expression is $(\sqrt{2} + 5)$.

Rationalize the denominator:

Ex. 4a $\frac{6}{5+\sqrt{7}}$ Ex. 4b $\frac{3+\sqrt{10}}{\sqrt{2}-5}$

Ex. 4c $\frac{\sqrt{a}-\sqrt{b}}{\sqrt{a}+\sqrt{b}}$, assume a and b are positive and a ≠ b.

Ex. 4d $\frac{2\sqrt{7}-\sqrt{6}}{\sqrt{6}-\sqrt{7}}$

Solution:

a) The conjugate of $5+\sqrt{7}$ is $5-\sqrt{7}$:

$\frac{6}{5+\sqrt{7}}$ (rationalize the denominator)

$= \frac{6}{(5+\sqrt{7})} \bullet \frac{(5-\sqrt{7})}{(5-\sqrt{7})}$ (simplify)

$= \frac{6(5-\sqrt{7})}{(5)^2-(\sqrt{7})^2} = \frac{30-6\sqrt{7}}{25-7}$

$= \frac{30-6\sqrt{7}}{18}$ (factor 6 out of the numerator)

$= \frac{6(5-\sqrt{7})}{18}$ (reduce)

$= \frac{5-\sqrt{7}}{3}$.

b) The conjugate of $\sqrt{2}-5$ is $\sqrt{2}+5$:

$\frac{3+\sqrt{10}}{\sqrt{2}-5}$ (rationalize the denominator)

$= \frac{(3+\sqrt{10})}{(\sqrt{2}-5)} \bullet \frac{(\sqrt{2}+5)}{(\sqrt{2}+5)}$ (simplify {use FOIL on the top})

$= \frac{3\sqrt{2}+3(5)+\sqrt{10}\bullet\sqrt{2}+\sqrt{10}(5)}{(\sqrt{2})^2-(5)^2}$

$= \frac{3\sqrt{2}+15+\sqrt{20}+5\sqrt{10}}{2-25}$ (but, $\sqrt{20} = \sqrt{2^2\bullet 5}$)

$= \frac{3\sqrt{2}+15+\sqrt{2^2\bullet 5}+5\sqrt{10}}{-23}$

$= \frac{3\sqrt{2}+15+2\sqrt{5}+5\sqrt{10}}{-23}$

Multiplying top and bottom by – 1, we get:

$= \frac{-15-3\sqrt{2}-2\sqrt{5}-5\sqrt{10}}{23}$.

c) The conjugate of $\sqrt{a} + \sqrt{b}$ is $\sqrt{a} - \sqrt{b}$:

$\dfrac{\sqrt{a}-\sqrt{b}}{\sqrt{a}+\sqrt{b}}$ (rationalize the denominator)

$= \dfrac{(\sqrt{a}-\sqrt{b})}{(\sqrt{a}+\sqrt{b})} \bullet \dfrac{(\sqrt{a}-\sqrt{b})}{(\sqrt{a}-\sqrt{b})}$ (simplify {use FOIL on the top})

$= \dfrac{\sqrt{a}\bullet\sqrt{a}-\sqrt{a}\bullet\sqrt{b}-\sqrt{b}\bullet\sqrt{a}+\sqrt{b}\bullet\sqrt{b}}{(\sqrt{a})^2-(\sqrt{b})^2}$

$= \dfrac{(\sqrt{a})^2-\sqrt{ab}-\sqrt{ab}+(\sqrt{b})^2}{a-b}$

$= \dfrac{a-2\sqrt{ab}+b}{a-b}$.

d) The conjugate of $\sqrt{6} - \sqrt{7}$ is $\sqrt{6} + \sqrt{7}$:

$\dfrac{2\sqrt{7}-\sqrt{6}}{\sqrt{6}-\sqrt{7}}$ (rationalize the denominator)

$= \dfrac{(2\sqrt{7}-\sqrt{6})}{(\sqrt{6}-\sqrt{7})} \bullet \dfrac{(\sqrt{6}+\sqrt{7})}{(\sqrt{6}+\sqrt{7})}$ (simplify {use FOIL on the top})

$= \dfrac{2\sqrt{7}\bullet\sqrt{6}+2\sqrt{7}\bullet\sqrt{7}-\sqrt{6}\bullet\sqrt{6}-\sqrt{6}\bullet\sqrt{7}}{(\sqrt{6})^2-(\sqrt{7})^2}$

$= \dfrac{2\sqrt{42}+2\sqrt{7^2}-\sqrt{6^2}-\sqrt{42}}{6-7}$

$= \dfrac{2\sqrt{42}+2(7)-(6)-\sqrt{42}}{6-7}$

$= \dfrac{8+\sqrt{42}}{-1} = -8 - \sqrt{42}$

<u>Simplify the following:</u>

Ex. 5a $\sqrt[3]{25} + \dfrac{4}{\sqrt[3]{5}}$

Ex. 5b $\dfrac{a+b}{\sqrt[3]{a}+\sqrt[3]{b}}$

<u>Solution:</u>

a) We will begin by rationalizing the denominator of $\dfrac{4}{\sqrt[3]{5}}$:

$\sqrt[3]{25} + \dfrac{4}{\sqrt[3]{5}} \bullet \dfrac{\sqrt[3]{5^2}}{\sqrt[3]{5^2}} = \sqrt[3]{25} + \dfrac{4\sqrt[3]{5^2}}{\sqrt[3]{5^3}} = \sqrt[3]{25} + \dfrac{4\sqrt[3]{5^2}}{5}$

$= \sqrt[3]{25} + \dfrac{4\sqrt[3]{25}}{5}$.

The two terms are like radicals so, we can combine them:

$$\sqrt[3]{25} + \frac{4\sqrt[3]{25}}{5} = \frac{\sqrt[3]{25}}{1} + \frac{4\sqrt[3]{25}}{5} = \frac{5\sqrt[3]{25}}{5} + \frac{4\sqrt[3]{25}}{5} = \frac{9\sqrt[3]{25}}{5}.$$

b) The conjugate of the denominator will not work since we are dealing with cube root inside of square roots. But, think of the formula for the sum of cubes: $(F + L)(F^2 - FL + L^2) = F^3 + L^3$. To get rid of the radicals, we will need to multiply top and the bottom by $F^2 - FL + L^2$, where $F = \sqrt[3]{a}$ and $L = \sqrt[3]{b}$:

$$F^2 - FL + L^2 = (\sqrt[3]{a})^2 - (\sqrt[3]{a})(\sqrt[3]{b}) + (\sqrt[3]{b})^2$$

$$= \sqrt[3]{a^2} - \sqrt[3]{ab} + \sqrt[3]{b^2}.$$

Thus, $\frac{a+b}{\sqrt[3]{a} + \sqrt[3]{b}}$

$$= \frac{(a+b)}{(\sqrt[3]{a} + \sqrt[3]{b})} \bullet \frac{(\sqrt[3]{a^2} - \sqrt[3]{ab} + \sqrt[3]{b^2})}{(\sqrt[3]{a^2} - \sqrt[3]{ab} + \sqrt[3]{b^2})}$$ (simplify the denominator)

$$= \frac{(a+b)(\sqrt[3]{a^2} - \sqrt[3]{ab} + \sqrt[3]{b^2})}{\sqrt[3]{a}(\sqrt[3]{a^2} - \sqrt[3]{ab} + \sqrt[3]{b^2}) + \sqrt[3]{b}(\sqrt[3]{a^2} - \sqrt[3]{ab} + \sqrt[3]{b^2})}$$

$$= \frac{(a+b)(\sqrt[3]{a^2} - \sqrt[3]{ab} + \sqrt[3]{b^2})}{\sqrt[3]{a}(\sqrt[3]{a^2}) - \sqrt[3]{a}(\sqrt[3]{ab}) + \sqrt[3]{a}(\sqrt[3]{b^2}) + \sqrt[3]{b}(\sqrt[3]{a^2}) - \sqrt[3]{b}(\sqrt[3]{ab}) + \sqrt[3]{b}(\sqrt[3]{b^2})}$$

$$= \frac{(a+b)(\sqrt[3]{a^2} - \sqrt[3]{ab} + \sqrt[3]{b^2})}{\sqrt[3]{a^3} - \sqrt[3]{a^2b} + \sqrt[3]{ab^2} + \sqrt[3]{a^2b} - \sqrt[3]{ab^2} + \sqrt[3]{b^3}}$$

$$= \frac{(a+b)(\sqrt[3]{a^2} - \sqrt[3]{ab} + \sqrt[3]{b^2})}{a - \sqrt[3]{a^2b} + \sqrt[3]{ab^2} + \sqrt[3]{a^2b} - \sqrt[3]{ab^2} + b}$$

$$= \frac{(a+b)(\sqrt[3]{a^2} - \sqrt[3]{ab} + \sqrt[3]{b^2})}{a+b}$$ (reduce)

$$= \sqrt[3]{a^2} - \sqrt[3]{ab} + \sqrt[3]{b^2}.$$

Concept #3 Rationalizing the Numerators.

There are several instances in Calculus where one has to rationalize the numerator instead of the denominator. The process works the same way except we need to get rid of the radicals in the numerator.

Rationalize the Numerator:

Ex. 6a $\dfrac{\sqrt{6}-2}{8}$ Ex. 6b $\dfrac{\sqrt{3+h}-\sqrt{3}}{h}$

Solution:

a) The conjugate of $\sqrt{6}$ – 2 is $\sqrt{6}$ + 2:

$\dfrac{\sqrt{6}-2}{8}$ (rationalize the numerator)

$= \dfrac{(\sqrt{6}-2)}{8} \bullet \dfrac{(\sqrt{6}+2)}{(\sqrt{6}+2)}$ (simplify)

$= \dfrac{(\sqrt{6}-2)(\sqrt{6}+2)}{8(\sqrt{6}+2)} = \dfrac{(\sqrt{6})^2-(2)^2}{8(\sqrt{6}+2)} = \dfrac{6-4}{8(\sqrt{6}+2)} = \dfrac{2}{8(\sqrt{6}+2)}$ (reduce)

$= \dfrac{1}{4(\sqrt{6}+2)}$ (distribute)

$= \dfrac{1}{4\sqrt{6}+8}$

b) The conjugate of $\sqrt{3+h}$ – $\sqrt{3}$ is $\sqrt{3+h}$ + $\sqrt{3}$:

$\dfrac{\sqrt{3+h}-\sqrt{3}}{h}$ (rationalize the numerator)

$= \dfrac{(\sqrt{3+h}-\sqrt{3})}{h} \bullet \dfrac{(\sqrt{3+h}+\sqrt{3})}{(\sqrt{3+h}+\sqrt{3})}$ (simplify)

$= \dfrac{(\sqrt{3+h})^2-(\sqrt{3})^2}{h(\sqrt{3+h}+\sqrt{3})} = \dfrac{3+h-3}{h(\sqrt{3+h}+\sqrt{3})} = \dfrac{h}{h(\sqrt{3+h}+\sqrt{3})}$ (reduce)

$= \dfrac{1}{\sqrt{3+h}+\sqrt{3}}$.

Notice that $\dfrac{\sqrt{3+h}-\sqrt{3}}{h}$ is undefined at h = 0. But, let's say we want to find the behavior of the expression when h ≈ 0. By rationalizing the numerator, the h divides out of the denominator. We can evaluate the new expression for h = 0 to find out the behavior of the original quotient when h ≈ 0:

$$\frac{1}{\sqrt{3+h}+\sqrt{3}} = \frac{1}{\sqrt{3+0}+\sqrt{3}} = \frac{1}{\sqrt{3}+\sqrt{3}} = \frac{1}{2\sqrt{3}}.$$

So, "near" h = 0, $\dfrac{\sqrt{3+h}-\sqrt{3}}{h} \approx \dfrac{1}{2\sqrt{3}}$.

This quantity represents the rate of change or ("the slope of the curve") of the square root function at x = 3 which you will study in Calculus.