Sect 11.7 - Radical Equations

Concept #1 Solutions to Radical Equations.

Definition of a Radical Equation
Any equation that has one or more radicals containing a variable is called a **Radical Equation**.

For example, $\sqrt[3]{x} = 4$ and $\sqrt{x} = 5$ are radical equations. To solve each equation, we first raise each side to the power of the index of the radical and solve:

$\sqrt[3]{x} = 4$ $\qquad\qquad$ $\sqrt{x} = 5$

$(\sqrt[3]{x})^3 = 4^3$ $\qquad\qquad$ $(\sqrt{x})^2 = 5^2$

But, if $\sqrt[n]{a}$ is a real number, then $(\sqrt[n]{a})^n = a$. So,

$(\sqrt[3]{x})^3 = 4^3$ $\qquad\qquad$ $(\sqrt{x})^2 = 5^2$

$x = 64$ $\qquad\qquad$ $x = 25$

Check

$\sqrt[3]{x} = 4$

$\sqrt[3]{64} = 4$?

$4 = 4$ True

Check

$\sqrt{x} = 5$

$\sqrt{25} = 5$?

$5 = 5$ True

The property we are using in each case is called the Power Property.

Power Property
If P and Q are algebraic expressions and n is a natural number, then every solution to the equation P = Q will also be a solution to the equation $P^n = Q^n$.

Note, the converse is not true. In other words, a solution to $P^n = Q^n$ is not necessarily a solution to P = Q. To see why, consider the following equation: $x = 5$ (now, square both sides)

$x^2 = 5^2$

$x^2 = 25$ (solve)

$x^2 – 25 = 0$

$(x – 5)(x + 5) = 0$

$x = 5$ or $x = – 5$

Clearly, $x = – 5$ is a false solution or an **extraneous solution**.

This means that whenever we use the power property and solve, we will need to check all the proposed answers in the original equation.

Concept #2 Solving a Radical Equation with One Radical.

Steps in Solving Radical Equations

1) Isolate a radical containing a variable on one side of the equation.
2) Raise both sides of the equation to the power of the index.
3) If the resulting equation still has a radical containing a variable, repeat steps #1 and #2. If not, solve the resulting equation.
4) Check all the proposed solutions in the original equation.

Solve the following:

Ex. 1a $\sqrt{u} - 6 = 7$ Ex. 1b $(3x - 4)^{1/2} + 7 = 9$

Ex. 1c $5 - \sqrt[3]{2x-7} = 9$ Ex. 1d $\sqrt{6x+7} - 1 = x + 1$

Ex. 1e $\sqrt{4x+5} - 13 = 2x - 18$

Ex. 1f $x - \sqrt{28-3x} = 0$ Ex. 1g $\sqrt[4]{5x^2+36} = x$

Solution:

a) $\sqrt{u} - 6 = 7$ (isolate $\sqrt{u}$)
$\sqrt{u} = 13$ (square both sides)
$(\sqrt{u})^2 = (13)^2$ (simplify)
$u = 169$
Check: $u = 169$
$\sqrt{u} - 6 = 7$
$\sqrt{169} - 6 = 7$?
$13 - 6 = 7$?
$7 = 7$ True
The solution is {169}.

b) $(3x - 4)^{1/2} + 7 = 9$ (isolate $(3x - 4)^{1/2}$)
$(3x - 4)^{1/2} = 2$ (square both sides)
$((3x - 4)^{1/2})^2 = (2)^2$ (simplify)
$3x - 4 = 4$ (solve)
$3x = 8$
$x = \frac{8}{3}$

Check: $x = \frac{8}{3}$

$(3x - 4)^{1/2} + 7 = 9$

$(3(\frac{8}{3}) - 4)^{1/2} + 7 = 9$?

$(8 - 4)^{1/2} + 7 = 9$?

$(4)^{1/2} + 7 = 9$?

$2 + 7 = 9$?

$9 = 9$ True

The solution is $\left\{\frac{8}{3}\right\}$.

c) $5 - \sqrt[3]{2x-7} = 9$ (isolate $\sqrt[3]{2x-7}$)

$-\sqrt[3]{2x-7} = 4$

$\sqrt[3]{2x-7} = -4$ (cube both sides)

$(\sqrt[3]{2x-7})^3 = (-4)^3$ (simplify)

$2x - 7 = -64$ (solve)

$2x = -57$

$x = -28.5$

Check: $x = -28.5$

$5 - \sqrt[3]{2x-7} = 9$

$5 - \sqrt[3]{2(-28.5)-7} = 9$?

$5 - \sqrt[3]{-57-7} = 9$?

$5 - \sqrt[3]{-64} = 9$?

$5 - (-4) = 9$?

$9 = 9$ True

The solution is $\{-28.5\}$.

d) $\sqrt{6x+7} - 1 = x + 1$ (isolate $\sqrt{6x+7}$)

$\sqrt{6x+7} = x + 2$ (square both sides)

$(\sqrt{6x+7})^2 = (x + 2)^2$ (simplify)

$6x + 7 = x^2 + 4x + 4$ (solve)

$0 = x^2 - 2x - 3$ (factor)

$0 = (x - 3)(x + 1)$

$x - 3 = 0$ or $x + 1 = 0$

$x = 3$ or $x = -1$

Check: $x = 3$

$\sqrt{6x+7} - 1 = x + 1$
$\sqrt{6(3)+7} - 1 = (3) + 1$?
$\sqrt{25} - 1 = (3) + 1$?
$5 - 1 = (3) + 1$?
$4 = 4$ True

$x = -1$

$\sqrt{6x+7} - 1 = x + 1$
$\sqrt{6(-1)+7} - 1 = (-1) + 1$?
$\sqrt{1} - 1 = (-1) + 1$?
$1 - 1 = (-1) + 1$?
$0 = 0$ True

The solutions are $\{-1, 3\}$.

e) $\sqrt{4x+5} - 13 = 2x - 18$ (isolate $\sqrt{4x+5}$)
$\sqrt{4x+5} = 2x - 5$ (square both sides)
$(\sqrt{4x+5})^2 = (2x - 5)^2$ (simplify)
$4x + 5 = 4x^2 - 20x + 25$ (solve)
$0 = 4x^2 - 24x + 20$ (factor)
$0 = 4(x^2 - 6x + 5)$
$0 = 4(x - 1)(x - 5)$
$4 = 0$ or $x - 1 = 0$ or $x - 5 = 0$
No Soln, or $x = 1$ or $x = 5$

Check: $x = 1$

$\sqrt{4x+5} - 13 = 2x - 18$
$\sqrt{4(1)+5} - 13 = 2(1) - 18$?
$\sqrt{9} - 13 = 2 - 18$?
$3 - 13 = 2 - 18$?
$-10 = -16$ False

$x = 5$

$\sqrt{4x+5} - 13 = 2x - 18$
$\sqrt{4(5)+5} - 13 = 2(5) - 18$?
$\sqrt{25} - 13 = 10 - 18$?
$5 - 13 = 10 - 18$?
$-8 = -8$ True

The solution is $\{5\}$.

f) $x - \sqrt{28-3x} = 0$ (isolate $\sqrt{28-3x}$)
$-\sqrt{28-3x} = -x$
$\sqrt{28-3x} = x$ (square both sides)
$(\sqrt{28-3x})^2 = (x)^2$ (simplify)
$28 - 3x = x^2$ (solve)
$0 = x^2 + 3x - 28$ (factor)
$0 = (x + 7)(x - 4)$
$x + 7 = 0$ or $x - 4 = 0$
$x = -7$ or $x = 4$

Check: $x = -7$ | $x = 4$

$x - \sqrt{28-3x} = 0$ | $x - \sqrt{28-3x} = 0$

$(-7) - \sqrt{28-3(-7)} = 0$? | $(4) - \sqrt{28-3(4)} = 0$?

$(-7) - \sqrt{49} = 0$? | $(4) - \sqrt{16} = 0$?

$-7 - 7 = 0$? | $4 - 4 = 0$?

$-14 = 0$ False | $0 = 0$ True

The solution is {4}.

g) $\sqrt[4]{5x^2+36} = x$ (raise both sides to the fourth power)

$(\sqrt[4]{5x^2+36})^4 = (x)^4$ (simplify)

$5x^2 + 36 = x^4$ (solve)

$x^4 - 5x^2 - 36 = 0$

This equation is in the form of a quadratic. Let $u = x^2$, then

$x^4 - 5x^2 - 36 = 0$

$(x^2)^2 - 5x^2 - 36 = 0$

$u^2 - 5u - 36 = 0$ (factor)

$(u - 9)(u + 4) = 0$

$u - 9 = 0$ or $u + 4 = 0$

Substitute x^2 back in for u:

$x^2 - 9 = 0$ or $x^2 + 4 = 0$

$(x - 3)(x + 3) = 0$ or No real solution

$x - 3 = 0$ or $x + 3 = 0$

$x = 3$ or $x = -3$

Check: $x = 3$ | $x = -3$

$\sqrt[4]{5x^2+36} = x$ | $\sqrt[4]{5x^2+36} = x$

$\sqrt[4]{5(3)^2+36} = (3)$? | $\sqrt[4]{5(-3)^2+36} = (-3)$?

$\sqrt[4]{45+36} = 3$? | $\sqrt[4]{45+36} = -3$?

$\sqrt[4]{81} = 3$? | $\sqrt[4]{81} = -3$?

$3 = 3$ True | $3 = -3$ False

The solution is {3}.

Concept #3 Solving Radical Equations Involving More than One Radical.

To solve a radical equation with more than one radical containing a variable, we simply isolate one radical and apply steps one and two of our procedure and then repeat the process for the other radical.

Solve the following:

Ex. 2a $\sqrt{4x+1} - \sqrt{x-1} = 2$ Ex. 2b $\sqrt[3]{5x^2-8x-2} - \sqrt[3]{x} = 0$

Ex. 2c $\sqrt{3\sqrt{2x+3}} - \sqrt{5x-6} = 0$ Ex. 2d $-\sqrt{x-7} = 2\sqrt{x-4} + 3$

Solution:

a) $\sqrt{4x+1} - \sqrt{x-1} = 2$ (isolate $\sqrt{4x+1}$)

$\sqrt{4x+1} = \sqrt{x-1} + 2$ (square both sides)

$(\sqrt{4x+1})^2 = (\sqrt{x-1} + 2)^2$ (simplify $\{(F + L)^2 = F^2 + 2FL + L^2\}$)

$4x + 1 = (\sqrt{x-1})^2 + 2(\sqrt{x-1})(2) + (2)^2$

$4x + 1 = (x - 1) + 4\sqrt{x-1} + 4$

$4x + 1 = x + 3 + 4\sqrt{x-1}$ (isolate $4\sqrt{x-1}$)

$3x - 2 = 4\sqrt{x-1}$ (square both sides)

$(3x - 2)^2 = (4\sqrt{x-1})^2$ (simplify $\{(F - L)^2 = F^2 - 2FL + L^2\}$)

$9x^2 - 12x + 4 = 16(x - 1)$

$9x^2 - 12x + 4 = 16x - 16$ (solve)

$9x^2 - 28x + 20 = 0$ (factor)

$(9x - 10)(x - 2) = 0$

$9x - 10 = 0$ or $x - 2 = 0$

$x = \frac{10}{9}$ or $x = 2$

Check: $x = \frac{10}{9}$ | $x = 2$

$\sqrt{4x+1} - \sqrt{x-1} = 2$ | $\sqrt{4x+1} - \sqrt{x-1} = 2$

$\sqrt{4(\frac{10}{9})+1} - \sqrt{\frac{10}{9}-1} = 2$? | $\sqrt{4(2)+1} - \sqrt{(2)-1} = 2$?

$\sqrt{\frac{40}{9}+1} - \sqrt{\frac{1}{9}} = 2$? | $\sqrt{9} - \sqrt{1} = 2$?

$\sqrt{\frac{49}{9}} - \sqrt{\frac{1}{9}} = 2$? | $3 - 1 = 2$?

$\frac{7}{3} - \frac{1}{3} = 2$? | $2 = 2$ True

$2 = 2$ True

The solutions are $\{\frac{10}{9}, 2\}$.

b) $\sqrt[3]{5x^2-8x-2} - \sqrt[3]{x} = 0$ (isolate $\sqrt[3]{5x^2-8x-2}$)

$\sqrt[3]{5x^2-8x-2} = \sqrt[3]{x}$ (cube both sides)

$\left(\sqrt[3]{5x^2 - 8x - 2}\right)^3 = \left(\sqrt[3]{x}\right)^3$ (simplify)

$5x^2 - 8x - 2 = x$ (solve)

$5x^2 - 9x - 2 = 0$ (factor)

$(5x + 1)(x - 2) = 0$

$5x + 1 = 0$ or $x - 2 = 0$

$x = -0.2$ or $x = 2$

Check: $x = -0.2$

$\sqrt[3]{5x^2 - 8x - 2} - \sqrt[3]{x} = 0$

$\sqrt[3]{5(-0.2)^2 - 8(-0.2) - 2} - \sqrt[3]{(-0.2)} = 0$?

$\sqrt[3]{5(0.04) - 8(-0.2) - 2} - \sqrt[3]{(-0.2)} = 0$?

$\sqrt[3]{0.2 + 1.6 - 2} - \sqrt[3]{-0.2} = 0$?

$\sqrt[3]{-0.2} - \sqrt[3]{-0.2} = 0$?

$0 = 0$ True

Check: $x = 2$

$\sqrt[3]{5x^2 - 8x - 2} - \sqrt[3]{x} = 0$

$\sqrt[3]{5(2)^2 - 8(2) - 2} - \sqrt[3]{(2)} = 0$?

$\sqrt[3]{5(4) - 8(2) - 2} - \sqrt[3]{2} = 0$?

$\sqrt[3]{20 - 16 - 2} - \sqrt[3]{2} = 0$?

$\sqrt[3]{2} - \sqrt[3]{2} = 0$?

$0 = 0$ True

The solutions are {– 0.2, 2}.

c) $\sqrt{3\sqrt{2x+3}} - \sqrt{5x-6} = 0$ (isolate $\sqrt{3\sqrt{2x+3}}$)

$\sqrt{3\sqrt{2x+3}} = \sqrt{5x-6}$ (square both sides)

$\left(\sqrt{3\sqrt{2x+3}}\right)^2 = \left(\sqrt{5x-6}\right)^2$ (simplify)

$3\sqrt{2x+3} = 5x - 6$ (square both sides)

$\left(3\sqrt{2x+3}\right)^2 = (5x - 6)^2$ (simplify $\{(F - L)^2 = F^2 - 2FL + L^2\}$)

$9(2x + 3) = (5x)^2 - 2(5x)(6) + (6)^2$

$18x + 27 = 25x^2 - 60x + 36$ (solve)

$0 = 25x^2 - 78x + 9$ (factor)

$0 = (25x - 3)(x - 3)$

$25x - 3 = 0$ or $x - 3 = 0$

$x = \frac{3}{25}$ $\qquad x = 3$

Check: $\quad x = \frac{3}{25}$

$\sqrt{3\sqrt{2x+3}} - \sqrt{5x-6} = 0$

$\sqrt{3\sqrt{2(\frac{3}{25})+3}} - \sqrt{5(\frac{3}{25})-6} = 0$?

$\sqrt{3\sqrt{\frac{6}{25}+3}} - \sqrt{\frac{3}{5}-6} = 0$?

But $\sqrt{\frac{3}{5}-6}$ is the square root of a negative number, which is undefined. Hence, the check fails.

Check: $\quad x = 3$

$\sqrt{3\sqrt{2x+3}} - \sqrt{5x-6} = 0$

$\sqrt{3\sqrt{2(3)+3}} - \sqrt{5(3)-6} = 0$?

$\sqrt{3\sqrt{9}} - \sqrt{9} = 0$?

$\sqrt{3\bullet 3} - \sqrt{9} = 0$?

$3 - 3 = 0$?

$0 = 0$ True

The solution is {3}.

d) $-\sqrt{x-7} = 2\sqrt{x-4} + 3$ (square both sides)

$(-\sqrt{x-7})^2 = (2\sqrt{x-4} + 3)^2$ (simplify $\{(F + L)^2 = F^2 + 2FL + L^2\}$)

$x - 7 = (2\sqrt{x-4})^2 + 2(2\sqrt{x-4})(3) + (3)^2$

$x - 7 = 4(x - 4) + 12\sqrt{x-4} + 9$

$x - 7 = 4x - 16 + 12\sqrt{x-4} + 9$

$x - 7 = 4x - 7 + 12\sqrt{x-4}$ (isolate $12\sqrt{x-4}$)

$-3x = 12\sqrt{x-4}$ (square both sides)

$(-3x)^2 = (12\sqrt{x-4})^2$

$9x^2 = 144(x - 4)$

$9x^2 = 144x - 576$ (solve)

$9x^2 - 144x + 576 = 0$ (factor)

$9(x^2 - 16x + 64) = 0$
$9(x - 8)^2 = 0$
$9 = 0$ or $x - 8 = 0$
No Soln or $x = 8$
Check: $x = 8$
$-\sqrt{x-7} = 2\sqrt{x-4} + 3$
$-\sqrt{(8)-7} = 2\sqrt{(8)-4} + 3$?
$-\sqrt{1} = 2\sqrt{4} + 3$?
$-1 = 2(2) + 3$?
$-1 = 7$ False
The solution is { }.

Concept #4 Applications

Solve the following:

Ex. 3 The approximate time t (in seconds) required for a pendulum to make one complete swing back and forth is given by $t \approx 2\pi\sqrt{\frac{L}{9.8}}$ where L is the length of the pendulum in meters.

a) Find the time (to the nearest tenth) it take a 0.8 meter pendulum to swing back and forth.

b) Find the length of a pendulum that takes 1.5 seconds to swing back and forth. (round to the nearest hundredth meter).

Solution:

a) Replace L by 0.8 meters and evaluate:

$t \approx 2\pi\sqrt{\frac{0.8}{9.8}} = 2\pi\sqrt{\frac{8}{98}} = 2\pi\sqrt{\frac{4}{49}} = 2\pi(\frac{2}{7}) = \frac{4}{7}\pi \approx 1.795... \approx 1.8$

It will take about 1.8 seconds.

b) Replace t by 1.5 and solve:

$1.5 \approx 2\pi\sqrt{\frac{L}{9.8}}$ (isolate $\sqrt{\frac{L}{9.8}}$)

$\frac{1.5}{2\pi} = \sqrt{\frac{L}{9.8}}$ (square both sides)

$\left(\frac{1.5}{2\pi}\right)^2 = \left(\sqrt{\frac{L}{9.8}}\right)^2$ (simplify)

$\frac{2.25}{4\pi^2} = \frac{L}{9.8}$ (solve)

$L = \frac{2.25 \bullet 9.8}{4\pi^2} \approx 0.558... \approx 0.56$ meters

The pendulum is approximately 0.56 meters.