Sect 11.8 - Complex Numbers

Concept #1 Imaginary Numbers

In the beginning of this chapter, we saw that the $\sqrt{-25}$ was undefined in the real numbers since there is no real number whose square is equal to a negative number. Also, we have seen in this chapter that many of our properties of radicals only work if $\sqrt[n]{a}$ is a real number. In Science and Engineering, such quantities like the $\sqrt{-25}$ occur all the time. So, we need to develop a number system that allows us to handle such numbers as the $\sqrt{-25}$. Since $\sqrt{-25}$ = $\sqrt{-1 \bullet 25}$, we are going to define a new type of number by letting the letter i represent $\sqrt{-1}$. Thus, $i^2 = -1$. Notice i is not a real number. Any number in the form bi where b is a non-zero real number is called an **imaginary number**.

Definition of the imaginary unit i

$i = \sqrt{-1}$

This implies that $i^2 = -1$.

Definition of imaginary numbers

Let b be any non-zero real number, then the set of imaginary numbers is the set of all numbers in the form bi, where $i = \sqrt{-1}$: {bi | b is a real number & b ≠ 0}. Note, this is different from the text.

Now, we have a way to define the square root of a negative number.

Definition of $\sqrt{-m}$ for m > 0

Let m be a real number such that m > 0, then $\sqrt{-m} = i\sqrt{m}$.

This definition will allow us to write a square root of negative number in terms of the square root of a positive number. Once we have done that, we can then use our property of radicals established previously to simplify.

Simplify the following:

Ex. 1a $\sqrt{-25}$

Ex. 1b $\sqrt{-1.21}$

Ex. 1c $\sqrt{-\frac{5}{9}}$

Ex. 1d $-\sqrt{-45}$

Solution:

a) $\sqrt{-25} = i\sqrt{25} = 5i$.

b) $\sqrt{-1.21} = i\sqrt{1.21} = 1.1i$.

c) $\sqrt{-\frac{5}{9}} = i\sqrt{\frac{5}{9}} = i\frac{\sqrt{5}}{\sqrt{9}} = \frac{\sqrt{5}}{3}i$.

d) $-\sqrt{-45} = -i\sqrt{45} = -i\sqrt{3^2 \bullet 5} = -3i\sqrt{5}$.

Before using any properties of radicals, we must use the definition of $\sqrt{-m}$ to first write the radicals in terms of i. Then, we can use our properties to simplify.

Simplify the following:

Ex. 2a $\frac{\sqrt{-144}}{\sqrt{-36}}$

Ex. 2b $\sqrt{-18} \bullet \sqrt{-2}$

Ex. 2c $\sqrt{-8} \bullet \frac{\sqrt{-27}}{\sqrt{-125}}$

Solution:

a) $\frac{\sqrt{-144}}{\sqrt{-36}} = \frac{i\sqrt{144}}{i\sqrt{36}} = \frac{12i}{6i} = 2$

b) $\sqrt{-18} \bullet \sqrt{-2} = i\sqrt{18} \bullet i\sqrt{2} = i^2\sqrt{36} = 6i^2$.

But, $i^2 = -1$, so $6i^2 = 6(-1) = -6$.

c) $\sqrt{-8} \bullet \frac{\sqrt{-27}}{\sqrt{-125}} = i\sqrt{8} \bullet \frac{i\sqrt{27}}{i\sqrt{125}} = i\sqrt{8} \bullet \frac{\sqrt{27}}{\sqrt{125}} = i\frac{\sqrt{8 \bullet 27}}{\sqrt{125}}$

$= i\frac{\sqrt{2^2 \bullet 3^2 \bullet 2 \bullet 3}}{\sqrt{5^2 \bullet 5}} = i\frac{2 \bullet 3\sqrt{2 \bullet 3}}{5\sqrt{5}} = \frac{6\sqrt{6}}{5\sqrt{5}}i$.

Now, we will need to rationalize the denominator:

$\frac{6\sqrt{6}}{5\sqrt{5}}i = \frac{6\sqrt{6}}{5\sqrt{5}} \bullet \frac{\sqrt{5}}{\sqrt{5}}i = \frac{6\sqrt{30}}{5\sqrt{5^2}}i = \frac{6\sqrt{30}}{5 \bullet 5}i = \frac{6\sqrt{30}}{25}i$.

A common mistake in example #2b is to multiply the – 18 and – 2 first: $\sqrt{-18} \bullet \sqrt{-2} \neq \sqrt{36} = 6$. We are not multiply two negative numbers, but two *square roots* of negative numbers. That is why we must apply the definition of $\sqrt{-m}$ first and write the numbers in terms of i.

Concept #2 Powers of i.

Since $i = \sqrt{-1}$ and $i^2 = -1$, we can derive the other powers of i:

$i = i$

$i^2 = -1$

$i^3 = i^2 \bullet i = -1 \bullet i = -i$

$i^4 = i^2 \bullet i^2 = -1 \bullet (-1) = 1$

$i^5 = i^4 \bullet i = 1 \bullet i = i$

$i^6 = i^4 \bullet i^2 = 1 \bullet (-1) = -1$

$i^7 = i^4 \bullet i^3 = 1 \bullet (-i) = -i$

$i^8 = (i^4)^2 = (1)^2 = 1$

$i^9 = (i^4)^2 \bullet i = 1 \bullet i = i$

$i^{10} = (i^4)^2 \bullet i^2 = 1 \bullet (-1) = -1$

$i^{11} = (i^4)^2 \bullet i^3 = 1 \bullet (-i) = -i$

$i^{12} = (i^4)^3 = (1)^3 = 1$

Notice that every multiple of four of the exponent, the pattern repeats. Thus, to simplify higher powers of i, we will divide the exponent by four. We will write the expression as i^4 raised to the quotient times i raised to the remainder. Since $i^4 = 1$, we only need to determine what i to the remainder is to get our answer.

Simplify the following:

Ex. 3a i^{93} Ex. 3b i^{246}

Ex. 3c i^{827} Ex. 3d i^{956}

Solution:

a) Since 93 ÷ 4 = 23 R 1, then
$i^{93} = (i^4)^{23} \bullet i = 1 \bullet i = i$.

b) Since 246 ÷ 4 = 61 R 2, then
$i^{246} = (i^4)^{61} \bullet i^2 = 1 \bullet (-1) = -1$.

c) Since 827 ÷ 4 = 206 R 3, then
$i^{827} = (i^4)^{206} \bullet i^3 = 1 \bullet (-i) = -i$.

d) Since 956 ÷ 4 = 239 R 0, then
$i^{956} = (i^4)^{239} = 1$.

Concept #3 Definition of a Complex Number

How do real numbers and imaginary numbers fit together? To see how, we need to define a larger set of numbers that will incorporate both. Such numbers are called **Complex Numbers**. A complex number is a number that contains a real number component and an imaginary number component. If the imaginary component is zero, then the complex number is just a real number. If the real component is zero and the imaginary component is not zero, then the complex number is just an imaginary number. If both the real component and the imaginary component are not zero, then the complex number is neither a pure imaginary nor a pure real number.

Definition of a Complex Number

A **complex number** is a number in the form a + bi where a and b are real numbers and $i = \sqrt{-1}$. This form, a + bi is referred to as standard form.

{a + bi | a and b are real numbers and $i = \sqrt{-1}$}

Notes:

1) If b = 0, then the complex number is a real number.
2) If b ≠ 0 and a = 0, then the complex number is an imaginary number (some books call this a pure imaginary number).
3) The number a represents the real part of the complex number. The number b represents the imaginary part of the complex number.
4) The complex number a – bi is the **complex conjugate** of a + bi.

Complex Numbers

Non-Real and Non-Imaginary Numbers (both a ≠ 0 and b ≠ 0) {a + bi | a and b are both non-zero real numbers}	
Real Numbers (b = 0) {a | a is a real number}	Imaginary Numbers (b ≠ 0, a = 0) {bi | b is a real number & b ≠ 0}

<u>Identify the real number part and the imaginary part of each number:</u>

Ex. 4a – 7 + 6i Ex. 4b – 8.1i Ex. 4c 9

Solution:

a) The real part is – 7 and the imaginary part is 6.

b) Since – 8.1i = 0 – 8.1i, the real part is 0 and the imaginary part is – 8.1.

c) Since 9 = 9 + 0i, The real part is 9 and the imaginary part is 0.

Concept #4 Addition, Subtraction, and Multiplication of Complex Numbers.

Addition and subtraction of complex numbers works a lot like simplifying algebraic expression. Pretend that i is a variable and combine like terms. Then write the answer in standard form.

Simplify:

Ex. 5a $(7 - 3i) + (-3 - 5i)$

Ex. 5b $(-2.1 + 1.4i) + (0.07 - 3i)$

Ex. 5c $\left(\frac{2}{3} - \frac{13}{7}i\right) - \left(\frac{7}{6} + \frac{3}{14}i\right)$

Ex. 5d $9.2 - (6 - 3.1i) - 4.2i$

Solution:

a) $(7 - 3i) + (-3 - 5i) = 7 - 3 - 3i - 5i = 4 - 8i$.

b) $(-2.1 + 1.4i) + (0.07 - 3i) = -2.1 + 0.07 + 1.4i - 3i$
$= -2.03 - 1.6i$.

c) $\left(\frac{2}{3} - \frac{13}{7}i\right) - \left(\frac{7}{6} + \frac{3}{14}i\right) = \frac{2}{3} - \frac{7}{6} - \frac{13}{7}i - \frac{3}{14}i$
$= \frac{4}{6} - \frac{7}{6} - \frac{26}{14}i - \frac{3}{14}i = -\frac{3}{6} - \frac{29}{14}i = -\frac{1}{2} - \frac{29}{14}i$.

d) $9.2 - (6 - 3.1i) - 4.2i = 9.2 - 6 + 3.1i - 4.2i = 3.2 - 1.1i$.

Multiplication of complex numbers works a lot like polynomials. The only steps we need to add are after multiplying, replace i^2 by – 1 and simplify and write the answer in standard form.

Simplify:

Ex. 6a $(-5i)(3i)$

Ex. 6b $7i(2 - 3i)$

Ex. 6c $(4 - 5i)(-2 + 6i)$

Ex. 6d $(8 - i)^2$

Ex. 6e $(-4 + 3i)^2$

Ex. 6f $(-3 + 7i)(-3 - 7i)$

Solution:

a) $(-5i)(3i) = -15i^2 = -15(-1) = 15.$

b) $7i(2-3i) = 14i - 21i^2 = 14i - 21(-1) = 21 + 14i.$

c) $(4-5i)(-2+6i) = -8 + 24i + 10i - 30i^2$
$= -8 + 34i - 30(-1) = -8 + 34i + 30 = 22 + 34i.$

d) $(8-i)^2 = 8^2 - 8i - 8i + i^2 = 64 - 16i + (-1) = 63 - 61i.$

e) $(-4+3i)^2 = (-4)^2 - 4(3i) - 4(3i) + (3i)^2 = 16 - 24i + 9i^2$
$= 16 - 24i + 9(-1) = 16 - 24i - 9 = 7 - 24i.$

f) $(-3+7i)(-3-7i) = (-3)^2 - (7i)^2 = 9 - 49i^2 = 9 - 49(-1)$
$= 9 + 49 = 58$

Concept #5 Division of Complex Numbers

When dividing by a complex number, we must make the denominator into a real number. Since $i = \sqrt{-1}$, we treat the division of complex numbers like we are rationalize the denominator of a radical expression. If the real part of the complex number in the denominator is zero, we multiply the top and the bottom by i to "rationalize the denominator" so to speak. Otherwise, we multiply the top and bottom by the complex conjugate of the denominator. In either case, we simplify and write the answer in standard form.

Simplify:

Ex. 7a $\frac{5-3i}{2i}$ Ex. 7b $\frac{3}{-4+7i}$

Ex. 7c $\frac{8+3i}{5-6i}$ Ex. 7d $\frac{11-2i}{11+2i}$

Solution:

a) Since the denominator, $2i$, has no real part, multiply top and bottom by i:

$$\frac{5-3i}{2i} = \frac{(5-3i)}{2i} \bullet \frac{i}{i} = \frac{5i-3i^2}{2i^2} = \frac{5i-3(-1)}{2(-1)} = \frac{5i+3}{-2} = -1.5 - 2.5i.$$

b) The complex conjugate of $-4+7i$ is $-4-7i$. Multiply top and bottom by $-4-7i$, and simplify:

$$\frac{3}{-4+7i} = \frac{3}{(-4+7i)} \bullet \frac{(-4-7i)}{(-4-7i)} = \frac{-12-21i}{(-4)^2-(7i)^2} = \frac{-12-21i}{16-49i^2} = \frac{-12-21i}{16-49(-1)}$$

$$= \frac{-12-21i}{16+49} = \frac{-12-21i}{65} = -\frac{12}{65} - \frac{21}{65}i.$$

c) The complex conjugate of $5 - 6i$ is $5 + 6i$. Multiply top and bottom by $5 + 6i$, and simplify:

$$\frac{8+3i}{5-6i} = \frac{(8+3i)}{(5-6i)} \bullet \frac{(5+6i)}{(5+6i)} = \frac{8\bullet5+8\bullet6i+3i\bullet5+3i\bullet6i}{(5)^2-(6i)^2} = \frac{40+48i+15i+18i^2}{25-36i^2}$$

$$= \frac{40+63i+18(-1)}{25-36(-1)} = \frac{40+63i-18}{25+36} = \frac{22+63i}{61} = \frac{22}{61} + \frac{63}{61}i.$$

d) The complex conjugate of $11 + 2i$ is $11 - 2i$. Multiply top and bottom by $11 - 2i$, and simplify:

$$\frac{11-2i}{11+2i} = \frac{(11-2i)}{(11+2i)} \bullet \frac{(11-2i)}{(11-2i)} = \frac{(11)^2-(2i)(11)-(2i)(11)+(2i)^2}{(11)^2-(2i)^2}$$

$$= \frac{121-22i-22i+4i^2}{121-4i^2} = \frac{121-44i+4(-1)}{121-4(-1)} = \frac{121-44i-4}{121+4} = \frac{117-44i}{125}$$

$$= \frac{117}{125} - \frac{44}{125}i.$$

Ex. 8a $\frac{-6+3\sqrt{-20}}{8}$ Ex. 8b $\frac{-18-\sqrt{-288}}{3}$

<u>Solution:</u>

a) $\frac{-6+3\sqrt{-20}}{8}$ (use the definition of $\sqrt{-m}$ and simplify)

$$= \frac{-6+3i\sqrt{20}}{8} = \frac{-6+3i\sqrt{2^2\bullet5}}{8} = \frac{-6+3i\bullet2\sqrt{5}}{8} = \frac{-6+6i\sqrt{5}}{8} \quad \text{(factor)}$$

$$= \frac{2(-3+3i\sqrt{5})}{8} \quad \text{(reduce)}$$

$$= \frac{-3+3i\sqrt{5}}{4} = -\frac{3}{4} + \frac{3\sqrt{5}}{4}i.$$

b) $\frac{-18-\sqrt{-288}}{3}$ (use the definition of $\sqrt{-m}$ and simplify)

$$= \frac{-18-i\sqrt{288}}{3} = \frac{-18-i\sqrt{12^2\bullet2}}{3} = \frac{-18-12i\sqrt{2}}{3} \quad \text{(factor)}$$

$$= \frac{3(-6-4i\sqrt{2})}{3} \quad \text{(reduce)}$$

$$= -6 - 4i\sqrt{2}.$$