# Sect 12.2 – The Quadratic Formula

Concept #1 The Derivation of the Quadratic Formula

Each time we complete the square and solve, we are following the same steps. The process can get quite involved particularly if we are working with less than friendly coefficients. So, we want to derive a formula for finding the solutions to a quadratic equation without having to complete the square each time. To do this, we will complete the square on the equation $ax^2 + bx + c = 0$, where $a \neq 0$. Afterwards, we will use the square root property and solve to get a formula for x in terms of the coefficients of a, b, and c. To help understand the steps, we will do a numerical example side by side the general derivation.

$3x^2 + 5x - 11 = 0$ $\qquad\qquad$ $ax^2 + bx + c = 0, \quad a \neq 0$

Isolate the variable terms on one side of the equation.

$3x^2 + 5x = 11$ $\qquad\qquad$ $ax^2 + bx = -c$

Divide both sides by the coefficient of the squared term and simplify.

$\frac{3x^2}{3} + \frac{5x}{3} = \frac{11}{3}$ $\qquad\qquad$ $\frac{ax^2}{a} + \frac{bx}{a} = -\frac{c}{a}$

$x^2 + \frac{5x}{3} = \frac{11}{3}$ $\qquad\qquad$ $x^2 + \frac{bx}{a} = -\frac{c}{a}$

$p = \frac{1}{2} \bullet \frac{5}{3} = \frac{5}{6}$. Add $\left(\frac{5}{6}\right)^2 = \frac{25}{36}$ to both sides and simplify. $\qquad$ $p = \frac{1}{2} \bullet \frac{b}{a} = \frac{b}{2a}$. Add $\left(\frac{b}{2a}\right)^2 = \frac{b^2}{4a^2}$ to both sides and simplify.

$x^2 + \frac{5x}{3} + \frac{25}{36} = \frac{11}{3} + \frac{25}{36}$ $\qquad\qquad$ $x^2 + \frac{bx}{a} + \frac{b^2}{4a^2} = -\frac{c}{a} + \frac{b^2}{4a^2}$

$x^2 + \frac{5x}{3} + \frac{25}{36} = \frac{132}{36} + \frac{25}{36}$ $\qquad\qquad$ $x^2 + \frac{bx}{a} + \frac{b^2}{4a^2} = -\frac{4ac}{4a^2} + \frac{b^2}{4a^2}$

$x^2 + \frac{5x}{3} + \frac{25}{36} = \frac{157}{36}$ $\qquad\qquad$ $x^2 + \frac{bx}{a} + \frac{b^2}{4a^2} = \frac{b^2 - 4ac}{4a^2}$

Rewrite in the form $(x + p)^2$.

$p = \frac{5}{6}$: $\left(x + \frac{5}{6}\right)^2 = \frac{157}{36}$ $\qquad\qquad$ $p = \frac{b}{2a}$: $\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$

Use the square root property.

$x + \frac{5}{6} = \pm\sqrt{\frac{157}{36}}$ $\qquad\qquad$ $x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$

But, $\pm\sqrt{\frac{157}{36}} = \pm\frac{\sqrt{157}}{\sqrt{36}} = \pm\frac{\sqrt{157}}{6}$ But, $\pm\sqrt{\frac{b^2-4ac}{4a^2}} = \pm\frac{\sqrt{b^2-4ac}}{\sqrt{4a^2}}$

$= \pm\frac{\sqrt{b^2-4ac}}{2|a|}$

So, $x + \frac{5}{6} = \pm\sqrt{\frac{157}{36}}$ becomes: So, $x + \frac{b}{2a} = \pm\sqrt{\frac{b^2-4ac}{4a^2}}$ becomes:

$x + \frac{5}{6} = \pm\frac{\sqrt{157}}{6}$ $x + \frac{b}{2a} = \pm\frac{\sqrt{b^2-4ac}}{2|a|}$

If a > 0, then $|a| = a$ and $\pm\frac{\sqrt{b^2-4ac}}{2|a|} = \pm\frac{\sqrt{b^2-4ac}}{2a}$. If a < 0. then $|a| = -a$ and $\pm\frac{\sqrt{b^2-4ac}}{2|a|} = \pm\frac{\sqrt{b^2-4ac}}{-2a} = \mp\frac{\sqrt{b^2-4ac}}{2a} = \pm\frac{\sqrt{b^2-4ac}}{2a}$

$x + \frac{5}{6} = \pm\frac{\sqrt{157}}{6}$ $x + \frac{b}{2a} = \pm\frac{\sqrt{b^2-4ac}}{2a}$

Now, solve for x.

$$x + \frac{5}{6} = \pm\frac{\sqrt{157}}{6} \qquad x + \frac{b}{2a} = \pm\frac{\sqrt{b^2-4ac}}{2a}$$

$$-\frac{5}{6} = -\frac{5}{6} \qquad -\frac{b}{2a} = -\frac{b}{2a}$$

$$x = -\frac{5}{6} \pm \frac{\sqrt{157}}{6} \qquad x = -\frac{b}{2a} \pm \frac{\sqrt{b^2-4ac}}{2a}$$

$$x = \frac{-5\pm\sqrt{157}}{6} \qquad x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

**The Quadratic Formula**

For any quadratic equation in the form $ax^2 + bx + c = 0$ where $a \neq 0$, then the solutions are:

$$x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

Thus, for $3x^2 + 5x - 11 = 0$, a = 3, b = 5, and c = – 11. The solutions are:

$$x = \frac{-(5)\pm\sqrt{(5)^2-4(3)(-11)}}{2(3)} = \frac{-(5)\pm\sqrt{25-4(3)(-11)}}{2(3)} = \frac{-5\pm\sqrt{25+132}}{6} = \frac{-5\pm\sqrt{157}}{6}.$$

Notice, just like the completing the square process, the quadratic formula will work regardless if the polynomial in the equation is factorable or not.

**<u>Solve the following:</u>**

Ex. 1a $2.1x^2 + 2.9x = 1$ Ex. 1b $5h^2 = 6(h - 5)$

Ex. 1c $3t(t + 1) = 7 - 2t$ Ex. 1d $27w^3 = -64$

Solution:

a) $2.1x^2 + 2.9x = 1$ (get zero on one side)

$2.1x^2 + 2.9x - 1 = 0$ (identify a, b, and c)

$a = 2.1$, $b = 2.9$, & $c = -1$ (plug into the quadratic formula)

$$x = \frac{-b\pm\sqrt{b^2-4ac}}{2a} = \frac{-(2.9)\pm\sqrt{(2.9)^2-4(2.1)(-1)}}{2(2.1)}$$

$$= \frac{-(2.9)\pm\sqrt{8.41-4(2.1)(-1)}}{2(2.1)} = \frac{-2.9\pm\sqrt{8.41+8.4}}{4.2} = \frac{-2.9\pm\sqrt{16.81}}{4.2}$$

$$= \frac{-2.9\pm4.1}{4.2}.$$

Thus, $x = \frac{-2.9-4.1}{4.2} = \frac{-7}{4.2} = \frac{-70}{42} = -\frac{5}{3}$ or

$x = \frac{-2.9+4.1}{4.2} = \frac{1.2}{4.2} = \frac{12}{42} = \frac{2}{7}$

The solutions are $\left\{-\frac{5}{3}, \frac{2}{7}\right\}$.

b) $5h^2 = 6(h - 5)$ (simplify)

$5h^2 = 6h - 30$ (get zero on one side)

$5h^2 - 6h + 30 = 0$ (identify a, b, and c)

$a = 5$, $b = -6$, & $c = 30$ (plug into the quadratic formula)

$$h = \frac{-b\pm\sqrt{b^2-4ac}}{2a} = \frac{-(-6)\pm\sqrt{(-6)^2-4(5)(30)}}{2(5)} = \frac{-(-6)\pm\sqrt{36-4(5)(30)}}{2(5)}$$

$$= \frac{6\pm\sqrt{36-600}}{10} = \frac{6\pm\sqrt{-564}}{10} = \frac{6\pm i\sqrt{564}}{10} = \frac{6\pm i\sqrt{2^2\bullet141}}{10}$$

$$= \frac{6\pm2i\sqrt{141}}{10} = \frac{2(3\pm i\sqrt{141})}{10} = \frac{3\pm i\sqrt{141}}{5}$$

The solutions are $\left\{\frac{3-i\sqrt{141}}{5}, \frac{3+i\sqrt{141}}{5}\right\}$.

c) $3t(t + 1) = 7 - 2t$ (simplify)

$3t^2 + 3t = 7 - 2t$ (get zero on one side)

$3t^2 + 5t - 7 = 0$ (identify a, b, and c)

$a = 3$, $b = 5$, & $c = -7$ (plug into the quadratic formula)

$$t = \frac{-b\pm\sqrt{b^2-4ac}}{2a} = \frac{-(5)\pm\sqrt{(5)^2-4(3)(-7)}}{2(3)} = \frac{-(5)\pm\sqrt{25-4(3)(-7)}}{2(3)}$$

$= \frac{-5 \pm \sqrt{25+84}}{6} = \frac{-5 \pm \sqrt{109}}{6}$

The solutions are $\left\{\frac{-5-\sqrt{109}}{6}, \frac{-5+\sqrt{109}}{6}\right\}$.

d) It may be tempting to solve for $w^3$ and then take the cube root of both sides. We would get $w = -\frac{4}{3}$. But, doing it this way, we would not find all the solutions.

$27w^3 = -64$ (get zero on one side)

$27w^3 + 64 = 0$ (factor: $F^3 + L^3 = (F + L)(F^2 - FL + L^2)$)

$(3w + 4)(9w^2 - 12w + 16) = 0$ (solve)

$3w + 4 = 0$

$w = -\frac{4}{3}$

or $9w^2 - 12w + 16 = 0$

$a = 9$, $b = -12$, & $c = 16$

$w = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-12) \pm \sqrt{(-12)^2 - 4(9)(16)}}{2(9)}$

$= \frac{-(-12) \pm \sqrt{144 - 4(9)(16)}}{2(9)} = \frac{12 \pm \sqrt{144 - 576}}{18} = \frac{12 \pm \sqrt{-432}}{18}$

$= \frac{12 \pm i\sqrt{432}}{18} = \frac{12 \pm i\sqrt{12^2 \bullet 3}}{18} = \frac{12 \pm 12i\sqrt{3}}{18} = \frac{6(2 \pm 2i\sqrt{3})}{18}$

$= \frac{2 \pm 2i\sqrt{3}}{3}$

The solutions are $\left\{-\frac{4}{3}, \frac{2-2i\sqrt{3}}{3}, \frac{2+2i\sqrt{3}}{3}\right\}$.

Concept #3 Applications

**<u>Set-up the equation and solve the following:</u>**

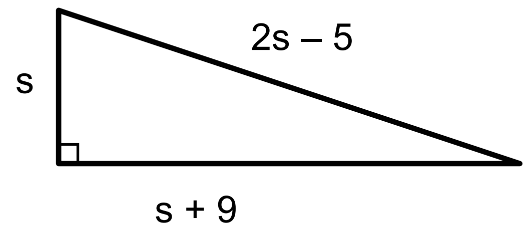
Ex. 2 The hypotenuse of a right triangle is five inches less than twice the length of the shorter leg. If the longer leg is nine inches longer than the shorter leg, find the dimension to the nearest tenth of an inch.

<u>Solution:</u>

Let s = the length of the shorter leg.

Since the hypotenuse is five inches less than twice the length of the shortest leg, then $2s - 5$ = the length of the hypotenuse.

Since the longer leg is nine inches longer than the shorter leg, then $s + 9$ = the length of the longer leg.




Recall the Pythagorean Theorem: $c^2 = a^2 + b^2$

c = 2s – 5, a = s and b = s + 9:

$c^2 = a^2 + b^2$ (substitute)

$(2s - 5)^2 = (s)^2 + (s + 9)^2$ (simplify)

$4s^2 - 20s + 25 = s^2 + s^2 + 18s + 81$

$4s^2 - 20s + 25 = 2s^2 + 18s + 81$ (get zero on one side)

$2s^2 - 38s - 56 = 0$

a = 2, b = – 38, & c = – 56

$$s = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-38) \pm \sqrt{(-38)^2 - 4(2)(-56)}}{2(2)}$$

$$= \frac{38 \pm \sqrt{1444 + 448}}{4} = \frac{38 \pm \sqrt{1892}}{4} = \frac{38 \pm \sqrt{2^2 \bullet 473}}{4}$$

$$= \frac{38 \pm 2\sqrt{473}}{4} = \frac{2(19 \pm \sqrt{473})}{4} = \frac{19 \pm \sqrt{473}}{2}$$

$s = \frac{19 - \sqrt{473}}{2} \approx -1.3742...$ Reject, the s cannot be negative.

or

$s = \frac{19 + \sqrt{473}}{2} = 20.3742... \approx 20.4$ inches

$s + 9 = \frac{19 + \sqrt{473}}{2} + 9 = 29.3742... \approx 29.4$ inches

$2s - 5 = 2(\frac{19 + \sqrt{473}}{2}) - 5 = 19 + \sqrt{473} - 5$

$= 35.7485... \approx 35.7$ inches

The dimensions are ≈ 20.4 in, ≈ 29.4 in, and ≈ 35.7 in.

Ex. 4 The braking distance (in feet) of a truck traveling at v mph is given by $d(v) = \frac{v^2}{16} + v$ $v \geq 0$.

How fast was the truck traveling (to the nearest mph) if it braking distance was 150 ft?

<u>Solution:</u>

d(v) = 150, so our equation is

$150 = \frac{v^2}{16} + v$ (multiply both sides by 16)

$16 \bullet 150 = 16(\frac{v^2}{16}) + 16v$

$2400 = v^2 + 16v$ (get zero on one side)

$0 = v^2 + 16v - 2400$

$a = 1$, $b = 16$, and $c = -2400$

$$v = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(16) \pm \sqrt{(16)^2 - 4(1)(-2400)}}{2(1)} = \frac{-16 \pm \sqrt{256 + 9600}}{2}$$

$$= \frac{-16 \pm \sqrt{9856}}{2} = \frac{-16 \pm \sqrt{8^2 \bullet 154}}{2} = \frac{-16 \pm 8\sqrt{154}}{2}$$

$$= -8 \pm 4\sqrt{154}$$

$v = -8 - 4\sqrt{154}$ = – number. Reject, $v > 0$.

or

$v = -8 + 4\sqrt{154} = 41.6386... \approx 42$ mph

The truck's speed was ≈ 42 mph.

Concept #4 The Discriminant and the Intercepts

Notice that the radicand $b^2 - 4ac$ in the quadratic formula determines the number and type of solution in the problem. We call the radicand $b^2 - 4ac$ the **discriminant**.

**The discriminant $b^2 - 4ac$**

Given a quadratic equation in the form $ax^2 + bx + c = 0$, $a \neq 0$, and a, b, and c are rational. The expression $b^2 - 4ac$ is called the **discriminant**.

1) If $b^2 - 4ac > 0$, then the equations will have two real solutions
   a) If $b^2 - 4ac$ is perfect square, then the real solutions will be rational.
   b) If $b^2 - 4ac$ is not a perfect square, then the real solutions will be irrational.
2) If $b^2 - 4ac < 0$, then the equation will have two non-real complex solutions that will be complex conjugates.
3) If $b^2 - 4ac = 0$, then the equation will have one real rational solution.

**<u>Use the discriminant to determine the number and type of solution:</u>**

Ex. 5a $5x^2 - 40x + 80 = 0$

Ex. 5b $0.9x^2 = -1.1x - 1.4$

Ex. 5c $2x(x - 2) = -1$

Ex. 5d $-3(x^2 + 6) = -17x + 2$

Ex. 5e $\sqrt{2}x^2 + 5x - 3\sqrt{2} = 0$

Solution:

a) $5x^2 - 40x + 80 = 0$ (use the discriminant)
$b^2 - 4ac = (-40)^2 - 4(5)(80) = 1600 - 1600 = 0$
So, the equation has one real rational solution.

b) $0.9x^2 = -1.1x - 1.4$ (get zero on one side)
$0.9x^2 + 1.1x + 1.4 = 0$ (use the discriminant)
$b^2 - 4ac = (1.1)^2 - 4(0.9)(1.4) = 1.21 - 5.04 = -3.83$
So, the equation has two non-real complex conjugate solutions.

c) $2x(x - 2) = -1$ (simplify)
$2x^2 - 4x = -1$ (get zero on one side)
$2x^2 - 4x + 1 = 0$ (use the discriminant)
$b^2 - 4ac = (-4)^2 - 4(2)(1) = 16 - 8 = 8$
But, 8 is not a perfect square.
So, the equation has two real irrational solutions.

d) $-3(x^2 + 6) = -17x + 2$ (simplify)
$-3x^2 - 18 = -17x + 2$ (get zero on one side)
$-3x^2 + 17x - 20 = 0$ (use the discriminant)
$b^2 - 4ac = (17)^2 - 4(-3)(-20) = 289 - 240 = 49$
But, 49 is a perfect square.
So, the equation has two real rational solutions.

e) $\sqrt{2}\,x^2 + 5x - 3\sqrt{2} = 0$
The discriminant test fails since $\sqrt{2}$ and $-3\sqrt{2}$ are not rational. The discriminant test only works if a, b, & c are rational.

Recall that the x-coordinates of the x-intercepts of a function are found by setting y = 0 and solving. If the function is a quadratic, the discriminant can be used to determine the number of x-intercepts the function has. So, let's recast example #5a through #5d and see how this works.

**Use the discriminant to determine the number of x-intercepts:**

Ex. 6a $f(x) = 5x^2 - 40x + 80$

Ex. 6b $g(x) = 0.9x^2 + 1.1x + 1.4$

Ex. 6c $h(x) = 2x^2 - 4x + 1$

Ex. 6d $r(x) = -3x^2 + 17x - 20$

Solution:

a) The discriminant = 0, so the function has one x-intercept.

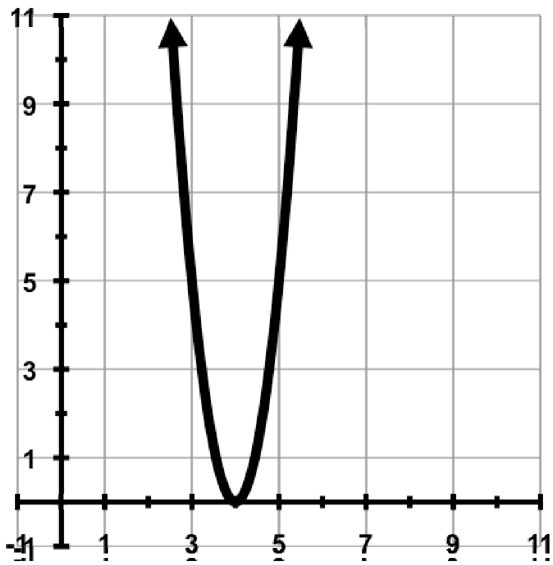
b) The discriminant < 0, so the function has no x-intercepts.

c) The discriminant > 0, so the function has two x-intercepts.

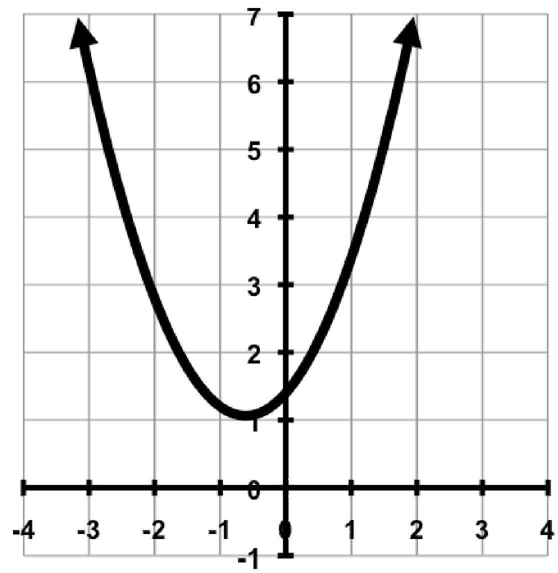
d) The discriminant > 0, so the function has two x-intercepts.

If we look at the graphs of the functions, we can see that how they match the information given by the discriminant.

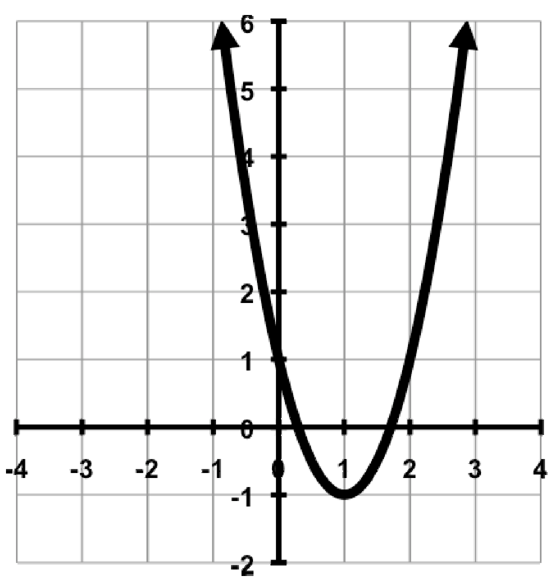
$f(x) = 5x^2 – 40x + 80$
has one x-intercept.
Notice the graph intersects the x-axis at (4, 0).



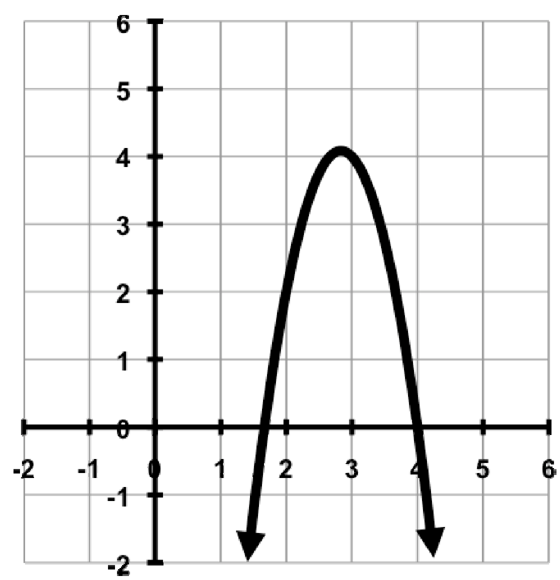

$g(x) = 0.9x^2 + 1.1x + 1.4$
has no x-intercepts.
Notice the graph does not intersect the x-axis.




$h(x) = 2x^2 – 4x + 1$
has two x-intercepts.
Notice the graph intersects the x-axis at (≈ 0.3, 0) and at (≈ 1.7, 0).




$r(x) = – 3x^2 + 17x – 20$
has no x-intercepts.
Notice the graph intersects the x-axis at (5/3, 0) and (4, 0).



**Find the intercepts of the following:**

Ex. 7a $f(x) = \frac{2}{3}x^2 + \frac{1}{4}x - 3$ Ex. 7b $g(x) = 4x^2 - 12x + 11$

Solution:

Recall how to find the intercepts:

1) We find the x-intercept(s) by replacing y by zero and solving for x.
2) We find the y-intercept(s) by replacing x by zero and solving for y.

a) To find the x-intercepts, replace y = f(x) by zero and solve:

$\frac{2}{3}x^2 + \frac{1}{4}x - 3 = 0$ (multiply both sides by 12)

$12\left(\frac{2}{3}x^2\right) + 12\left(\frac{1}{4}x\right) - 12(3) = 12(0)$

$8x^2 + 3x - 36 = 0$ (use the quadratic formula)

a = 8, b = 3, and c = – 36

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(3) \pm \sqrt{(3)^2 - 4(8)(-36)}}{2(8)} = \frac{-3 \pm \sqrt{9+1152}}{16}$$

$$= \frac{-3 \pm \sqrt{1161}}{16} = \frac{-3 \pm \sqrt{3^2 \bullet 129}}{16} = \frac{-3 \pm 3\sqrt{129}}{16}.$$

The x-intercepts are $\left(\frac{-3-3\sqrt{129}}{16}, 0\right)$ and $\left(\frac{-3+3\sqrt{129}}{16}, 0\right)$.

To find the y-intercepts, replace x by zero and solve:

$f(0) = \frac{2}{3}(0)^2 + \frac{1}{4}(0) - 3 = -3$

The y-intercept is (0, – 3).

b) To find the x-intercepts, replace y = g(x) by zero and solve:

$4x^2 - 12x + 11 = 0$ (use the quadratic formula)

a = 4, b = – 12, and c = 11

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-12) \pm \sqrt{(-12)^2 - 4(4)(11)}}{2(4)} = \frac{12 \pm \sqrt{144-176}}{8}$$

$$= \frac{12 \pm \sqrt{-32}}{8} = \frac{12 \pm i\sqrt{4^2 \bullet 2}}{8} = \frac{12 \pm 4i\sqrt{2}}{8} = \frac{3 \pm i\sqrt{2}}{2}.$$ Since the solutions are complex conjugates, there are no x-intercepts.

To find the y-intercepts, replace x by zero and solve:

$f(0) = 4(0)^2 - 12(0) + 11 = 11$

The y-intercept is (0, 11).

Concept #5 Mixed Review: Methods to Solve a Quadratic Equation

We have three methods for solving quadratic equations:

1) Solve by factoring.
2) Solve by using the square root property with a perfect square.
3) Solve by using the quadratic formula.

Solving by factoring only works if the problem is factorable. We use this method when the polynomial is easily factored. Otherwise, we will use one of the other methods. Solving by using the square root property with a perfect square we usually use if we already have a perfect square in the problem that incorporates all the variables. If we do not have a perfect square that incorporates all the variables or if the problem is not easily factorable, then we use the quadratic formula.

**Solve the following:**

| | | | |
|---|---|---|---|
| Ex. 8a | $(2x + 5)^2 - 11 = -5$ | Ex. 8b | $(3x + 1)^2 + 4 = -5x$ |
| Ex. 8c | $2x(3x - 4) = 5x^2 + 9$ | Ex. 8d | $-3x(x - 6) - 5 = 27$ |

Solution:

a) We have a perfect square that incorporates all the variables so, we will isolate the perfect square and use the square root property.

$(2x + 5)^2 - 11 = -5$

$(2x + 5)^2 = 6$ (use the square root property)

$(2x + 5) = \pm\sqrt{6}$ (solve)

$2x = -5 \pm \sqrt{6}$

$x = \frac{-5\pm\sqrt{6}}{2}$

So, the solutions are $\left\{\frac{-5-\sqrt{6}}{2}, \frac{-5+\sqrt{6}}{2}\right\}$.

b) Our perfect square does not incorporate all the variables, so we will need to pick a different method. Let's first simplify:

$(3x + 1)^2 + 4 = -5x$ (expand and simplify)

$9x^2 + 6x + 1 + 4 = -5x$

$9x^2 + 6x + 5 = -5x$ (get zero on one side)

$9x^2 + 11x + 5 = 0$

This does not easily factor so, let's use the quadratic formula.

$9x^2 + 11x + 5 = 0$

$a = 9$, $b = 11$, and $c = 5$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(11) \pm \sqrt{(11)^2 - 4(9)(5)}}{2(9)} = \frac{-11 \pm \sqrt{121 - 180}}{18}$$

$$= \frac{-11 \pm \sqrt{-59}}{18} = \frac{-11 \pm i\sqrt{59}}{18}$$

So, the solutions are $\left\{\frac{-11 - i\sqrt{59}}{18}, \frac{-11 + i\sqrt{59}}{18}\right\}$.

c) $2x(3x - 4) = 5x^2 + 9$ (simplify)
$6x^2 - 8x = 5x^2 + 9$ (get zero on one side)
$x^2 - 8x - 9 = 0$
This polynomial easily factors:
$(x - 9)(x + 1) = 0$
$x - 9 = 0$ or $x + 1 = 0$
$x = 9$ or $x = -1$
So, the solutions are $\{-1, 9\}$

d) $-4x(x - 6) + 5 = -26$ (simplify)
$-4x^2 + 24x + 5 = -26$ (get zero on one side)
$-4x^2 + 24x + 31 = 0$
This does not easily factor so, let's use the quadratic formula.
$-4x^2 + 24x + 31 = 0$
$a = -4$, $b = 24$, and $c = 31$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(24) \pm \sqrt{(24)^2 - 4(-4)(31)}}{2(-4)} = \frac{-24 \pm \sqrt{576 + 496}}{-8}$$

$$= \frac{-24 \pm \sqrt{1072}}{-8} = \frac{-24 \pm \sqrt{4^2 \bullet 67}}{-8} = \frac{-24 \pm 4\sqrt{67}}{-8}$$

$$= \frac{-4(6 \mp \sqrt{67})}{-8} = \frac{6 \mp \sqrt{67}}{2} = \frac{6 \pm \sqrt{67}}{2}.$$

Thus, the solutions are $\left\{\frac{6 - \sqrt{67}}{2}, \frac{6 + \sqrt{67}}{2}\right\}$.