Sect 12.5 - Vertex of a Parabola and Applications

Concept #1 Writing a Quadratic Function in Graphing Form

Given a quadratic function in the form $f(x) = ax^2 + bx + c$ where $a \neq 0$, we want to get the equation into graphing form $f(x) = a(x - h)^2 + k$ so that we can easily sketch the graph. To do this, we can complete the square in a similar fashion as we did in solving quadratic equations. We will have to make some adjustment to our procedure since we will only be working with one side of the equation.

Procedure for converting from $f(x) = ax^2 + bx + c$ to $f(x) = a(x - h)^2 + k$

1) If the coefficient of the squared term is not one, factor it out from both the squared term and the linear term. The coefficient of the linear term inside the parenthesis will be b divided by a.
2) Divide the coefficient of the linear term by 2 (or multiply it by a half) to get p. Square p and write plus the result (p^2) and minus the result (p^2) inside of the set of parenthesis.
3) Move the minus p^2 out of the parenthesis by distributing a to that value. Combine the result with c.
4) Rewrite the perfect square in the parenthesis as $(x + p)^2$. The quadratic function should be in graphing form.

Find the vertex, axis of symmetry, and the maximum or minimum value of the following:

Ex. 1 $f(x) = 3x^2 + 24x - 5$

Solution:

Let's convert the function into graphing form:

$f(x) = 3x^2 + 24x - 5$ (factor out 3 from $3x^2 + 12x$)

$f(x) = 3(x^2 + 8x) - 5$ ($p = 8 \div 2 = 4$. Add and subtract $(4)^2 = 16$)

$f(x) = 3(x^2 + 8x \mathbf{+ 16 - 16}) - 5$ (move the $- 16$ out and times it by 3)

$f(x) = 3(x^2 + 8x + 16) - 48 - 5$ (combine $- 48$ and $- 5$)

$f(x) = 3(x^2 + 8x + 16) - 53$ (rewrite in the form $(x + p)^2$)

$f(x) = 3(x + 4)^2 - 53$ $a = 3$, $h = - 4$, and $k = - 53$

Vertex: $(- 4, - 53)$

Axis of symmetry: $x = - 4$

Since $a > 0$, the graph opens upward, so f(x) has a minimum value of $- 53$ at $x = - 4$.

Ex. 2 $g(x) = -2x^2 + 5x - 7$

Solution:

Let's convert the function into graphing form:

$g(x) = -2x^2 + 5x - 7$ (factor out – 2 from $-2x^2 + 5x$)

$g(x) = -2(x^2 - \frac{5}{2}x) - 7$ ($p = -\frac{5}{2} \bullet \frac{1}{2} = -\frac{5}{4}$. Add and subtract $\left(-\frac{5}{4}\right)^2 = \frac{25}{16}$)

$g(x) = -2(x^2 - \frac{5}{2}x + \frac{25}{16} - \frac{25}{16}) - 7$ (move the $-\frac{25}{16}$ out and times it by – 2)

$g(x) = -2(x^2 - \frac{5}{2}x + \frac{25}{16}) + \frac{25}{8} - \frac{56}{8}$ (combine $\frac{25}{8}$ and $-\frac{56}{8}$)

$g(x) = -2(x^2 - \frac{5}{2}x + \frac{25}{16}) - \frac{31}{8}$ (rewrite in the form $(x + p)^2$)

$g(x) = -2(x - \frac{5}{4})^2 - \frac{31}{8}$ $a = -2$, $h = \frac{5}{4}$, and $k = -\frac{31}{8}$

Vertex: $\left(\frac{5}{4}, -\frac{31}{8}\right)$

Axis of symmetry: $x = \frac{5}{4}$

Since $a < 0$, the graph opens downward, so g(x) has a maximum value of $-\frac{31}{8}$ at $x = \frac{5}{4}$.

<u>Given $p(x) = 4x^2 - 16x + 13$</u>

Ex. 3 a) Write the function in graphing form.

b) Find the vertex, axis of symmetry, and the maximum or minimum value.

c) Find the x- and y-intercepts.

d) Sketch the graph.

Solution:

a) $p(x) = 4x^2 - 16x + 13$ (factor out 4 from $4x^2 + 16x$)

$p(x) = 4(x^2 - 4x) + 13$ ($p = -4 \div 2 = -2$. Add and subtract $(-2)^2 = 4$)

$p(x) = 4(x^2 - 4x \mathbf{+ 4 - 4}) + 13$ (move the – 4 out and times it by 4)

$p(x) = 4(x^2 - 4x + 4) - 16 + 13$ (combine – 16 and 13)

$p(x) = 4(x^2 - 4x + 4) - 3$ (rewrite in the form $(x + p)^2$)

$p(x) = 4(x - 2)^2 - 3$ $a = 4$, $h = 2$, and $k = -3$

b) Vertex: (2, – 3)

Axis of symmetry: x = 2

Since $a > 0$, the graph opens upward, so p(x) has a minimum value of – 3 at x = 2.

c) x-intercepts. Let p(x) = 0:

$0 = 4x^2 - 16x + 13$

a = 4, b = – 16, and c = 13

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-16) \pm \sqrt{(-16)^2 - 4(4)(13)}}{2(4)} = \frac{16 \pm \sqrt{256 - 208}}{8} = \frac{16 \pm \sqrt{48}}{8}$$

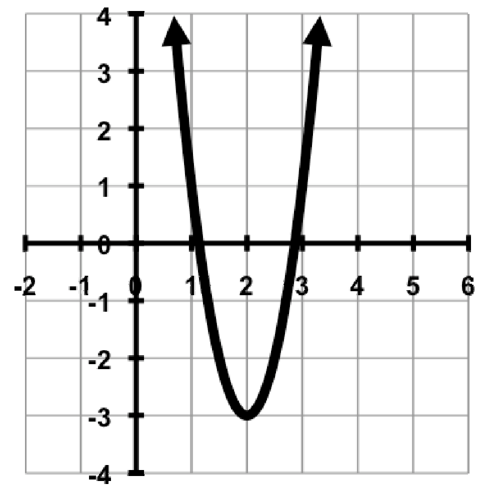
$$= \frac{16 \pm \sqrt{4^2 \bullet 3}}{8} = \frac{16 \pm 4\sqrt{3}}{8} = \frac{4(4 \pm \sqrt{3})}{8} = \frac{4 \pm \sqrt{3}}{2}$$

The x-intercepts are $\left(\frac{4 - \sqrt{3}}{2}, 0\right)$ and $\left(\frac{4 + \sqrt{3}}{2}, 0\right)$

y-intercepts. Let x = 0:

$p(0) = 4(0)^2 - 16(0) + 13 = 13$

The y-intercept is (0, 13).



d) Let's go through the steps of our general strategy :

i) Since | a | = 4, the graph is stretched by a factor of 4.

ii) Since a is positive, the graph is not reflected across the x-axis.

iii) Since k is – 3 and h is 2, the graph is shifted down by 3 units and to the right by 2 unit.

Concept #2 Vertex Formula

Given a quadratic function in the form $f(x) = ax^2 + bx + c$ where $a \neq 0$, we want to derive a formula for getting the function into graphing form much in the same way as we derived the quadratic formula. To write the function in graphing form ($f(x) = a(x - h)^2 + k$), we need to find a, h, and k. The value of a is the same value of a in the standard form of the quadratic equation. If we know h, we can find k by evaluating the function at x = h since $f(\mathbf{h}) = a((\mathbf{h}) - h)^2 + k = a(0)^2 + k = k$.

Thus, k = f(h).

So, let's find a formula for h. We will also do a numerical example to see how the steps work:

$f(x) = 3x^2 + 5x - 7$ $\qquad$ $f(x) = ax^2 + bx + c$

Factor out a from the variable terms.

$f(x) = 3(x^2 + \frac{5}{3}x) - 7$ $\qquad$ $f(x) = a(x^2 + \frac{b}{a}x) + c$

$p = \frac{1}{2} \bullet \frac{5}{3} = \frac{5}{6}$. $\left(\frac{5}{6}\right)^2 = \frac{25}{36}$ $\qquad$ $p = \frac{1}{2} \bullet \frac{b}{a} = \frac{b}{2a}$. $\left(\frac{b}{2a}\right)^2 = \frac{b^2}{4a^2}$

Add and subtract the result inside the parenthesis.

$f(x) = 3(x^2 + \frac{5}{3}x + \frac{25}{36} - \frac{25}{36}) - 7$ $\qquad$ $f(x) = a(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} - \frac{b^2}{4a^2}) + c$

Move the negative constant term out and times it by a.

$f(x) = 3(x^2 + \frac{5}{3}x + \frac{25}{36}) - 3(\frac{25}{36}) - 7$ $\qquad$ $f(x) = a(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}) - a(\frac{b^2}{4a^2}) + c$

$f(x) = 3(x^2 + \frac{5}{3}x + \frac{25}{36}) - \frac{25}{12} - \frac{84}{12}$ $\qquad$ $f(x) = a(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}) - \frac{b^2}{4a} + \frac{4ac}{4a}$

Combine the constant terms outside of the parenthesis.

$f(x) = 3(x^2 + \frac{5}{3}x + \frac{25}{36}) - \frac{109}{12}$ $\qquad$ $f(x) = a(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}) + \frac{4ac - b^2}{4a}$

Rewrite in the perfect square trinomial in the form $(x + p)^2$.

$f(x) = 3(x + \frac{5}{6})^2 - \frac{109}{12}$ $\qquad$ $f(x) = a(x + \frac{b}{2a})^2 + \frac{4ac - b^2}{4a}$

$f(x) = 3(x - (-\frac{5}{6}))^2 - \frac{109}{12}$ $\qquad$ $f(x) = a(x - (-\frac{b}{2a}))^2 + \frac{4ac - b^2}{4a}$

Thus, $h = -\frac{5}{6}$ $\qquad$ Thus, $h = -\frac{b}{2a}$

Incidentally, we do have a formula for k: $k = \frac{4ac - b^2}{4a}$, but it is usually easier to use $k = f(h)$ to find k. Notice that the formula $h = -\frac{b}{2a}$ looks like the quadratic formula without the ± radical.

Vertex Formula

Given $f(x) = ax^2 + bx + c$ where $a \neq 0$, the vertex (h, k) is $\left(-\frac{b}{2a}, f(-\frac{b}{2a})\right)$.

Concept #3 Determining the Intercepts and the Vertex of a Quadratic Function

For the following quadratic functions:

a) Find the vertex, the axis of symmetry, and the minimum or maximum value.
b) Find the x- and y-intercepts.
c) Write the function in graphing form.
d) Sketch the graph.

Ex. 5 $\qquad f(x) = 2x^2 + 12x + 17$

Solution:

a) $f(x) = 2x^2 + 12x + 17$, a = 2, b = 12, & c = 17 (vertex formula)

$h = -\frac{b}{2a} = -\frac{(12)}{2(2)} = -3$

$k = f(-3) = 2(-3)^2 + 12(-3) + 17 = 18 - 36 + 17 = -1$

Thus, the vertex is (– 3, – 1).

The axis of symmetry is x = – 3.

Since a > 0, f has a minimum value of – 1 at x = – 3.

b) x-intercepts. Let f(x) = 0:

$0 = 2x^2 + 12x + 17$, a = 2, b = 12, & c = 17 (quadratic formula)

$$x = \frac{-b\pm\sqrt{b^2-4ac}}{2a} = \frac{-(12)\pm\sqrt{(12)^2-4(2)(17)}}{2(2)} = \frac{-12\pm\sqrt{144-136}}{4}$$

$$= \frac{-12\pm\sqrt{8}}{4} = \frac{-12\pm\sqrt{2^2\bullet 2}}{4} = \frac{-12\pm 2\sqrt{2}}{4} = \frac{2(-6\pm\sqrt{2})}{4}$$

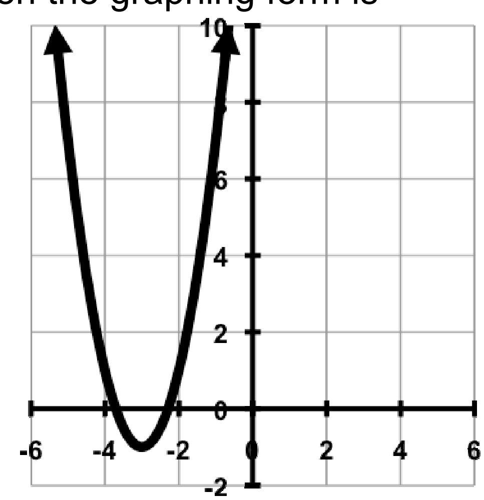
$$= \frac{-6\pm\sqrt{2}}{2}.$$

y - intercepts. Let x = 0:

$f(0) = 2(0)^2 + 12(0) + 17 = 17$

So, the x-intercepts are $\left(\frac{-6-\sqrt{2}}{2}, 0\right)$ and $\left(\frac{-6+\sqrt{2}}{2}, 0\right)$ and the y-intercept is (0, 17).

c) Since a = 2, h = – 3, and k = – 1, then the graphing form is $f(x) = 2(x + 3)^2 - 1$



d) Let's go through the steps of our general strategy:

i) Since $|a| = 2$, the graph is stretched by a factor of 2.

ii) Since a is positive, the graph is not reflected across the x-axis.

iii) Since k is – 1 and h is – 3, the graph is shifted down by 1 unit and to the left by 3 unit.

Ex. 6 $g(x) = -\frac{1}{3}x^2 + 4x - 14$

<u>Solution:</u>

a) $g(x) = -\frac{1}{3}x^2 + 4x - 14$, $a = -\frac{1}{3}$, $b = 4$, & $c = -14$

(vertex formula)

$$h = -\frac{b}{2a} = -\frac{(4)}{2(-\frac{1}{3})} = -4 \div \left(-\frac{2}{3}\right) = -\frac{4}{1}\bullet\left(-\frac{3}{2}\right) = 6$$

$$k = g(6) = -\frac{1}{3}(6)^2 + 4(6) - 14 = -12 + 24 - 14 = -2$$

Thus, the vertex is (6, – 2).
The axis of symmetry is x = 6.
Since a < 0, f has a maximum value of – 2 at x = 6.

b) x-intercepts. Let f(x) = 0:

$0 = -\frac{1}{3}x^2 + 4x - 14$, $a = -\frac{1}{3}$, $b = 4$, & $c = -14$

(quadratic formula)

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(4) \pm \sqrt{(4)^2 - 4(-\frac{1}{3})(-14)}}{2(-\frac{1}{3})} = \frac{-4 \pm \sqrt{16 - \frac{56}{3}}}{-\frac{2}{3}}$$

$$= \frac{-4 \pm \sqrt{\frac{48}{3} - \frac{56}{3}}}{-\frac{2}{3}} = \frac{-4 \pm \sqrt{-\frac{8}{3}}}{-\frac{2}{3}}$$, but this is not a real number.

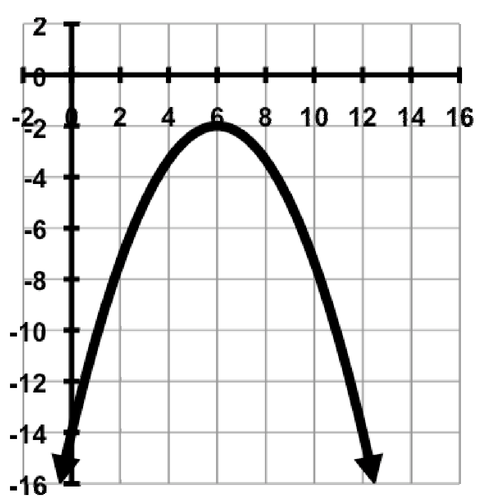
y - intercepts. Let x = 0:

$$f(0) = -\frac{1}{3}(0)^2 + 4(0) - 14 = -14$$

So, there no x-intercepts.
y-intercept is (0, – 14).

c) Since $a = -\frac{1}{3}$, h = 6, and k = – 2, then the graphing form is

$$f(x) = -\frac{1}{3}(x - 6)^2 - 2$$



d) Let's go through the steps of our general strategy:

i) Since $|a| = \frac{1}{3}$, the graph is shrunk by a factor of $\frac{1}{3}$.

ii) Since a is negative, the graph is reflected across the x-axis.

iii) Since k is – 2 and h is 6, the graph is shifted down by 2 units and to the right by 6 units.

Solve the following:

Ex. 7 A manufacture of wide screen TV sets determines that their profits (p(x) is dollars) depends on the number of TV sets, x, that they produce and sell and is given by:

$p(x) = -0.15x^2 + 45x - 1450$ where $x \geq 0$

a) Find the y-intercept and interpret its meaning.

b) How many TV sets must be produced and sold for the manufacture to break even?

c) How many TV sets must be produced and sold to maximize the profit? What is the maximum profit?

d) Sketch the graph of the profit function.

Solution:

a) y-intercept. Let x = 0:

$p(0) = -0.15(0)^2 + 45(0) - 1450 = -1450$

So, the y-intercept is (0, – 1450).

This means that is the manufacture produces no TV sets, then the manufacture will lose $1450.

b) To break even means the profit is 0. Setting the p(x) = 0 and solving yields:

$-0.15x^2 + 45x - 1450 = 0$, $a = -0.15$, $b = 45$, & $c = -1450$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(45) \pm \sqrt{(45)^2 - 4(-0.15)(-1450)}}{2(-0.15)}$$

$$= \frac{-45 \pm \sqrt{2025 - 870}}{-0.3} = \frac{-45 \text{Š} \sqrt{1155}}{-0.3}$$. We will need to approximate these solutions and round to the nearest whole number since only whole TV sets can be produced and sold.

So, $x = \frac{-45 - \sqrt{1155}}{-0.3} \approx 263$ or $x = \frac{-45 - \sqrt{1155}}{-0.3} \approx 37$

Either they can produce and sell 37 TV sets or 263 TV sets.

c) Since a < 0, the maximum profit will occur at the vertex.

$h = -\frac{b}{2a} = -\frac{(45)}{2(-0.15)} = 150$

$k = p(150) = -0.15(150)^2 + 45(150) - 1450$

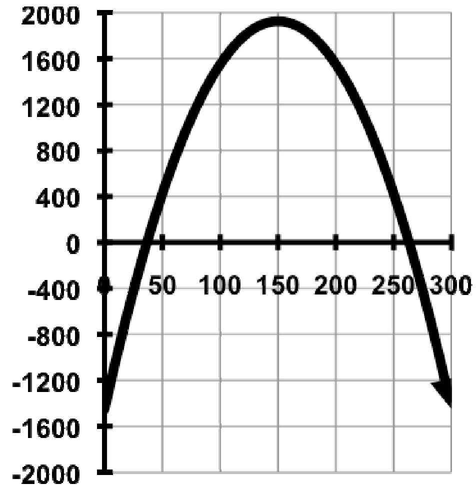
$= -3375 + 6750 - 1450 = 1925$

Thus, the vertex is (150, 1925).

The manufacture will have a maximum profit of $1925 when 150 TV sets are produced and sold.

d) Since $a = -0.15$, $h = 150$, and $k = 1925$, then the graphing form is $p(x) = -0.15(x - 150)^2 + 1925$

Let's go through the steps of our general strategy:



i) Since $|a| = 0.15$, the graph is shrunk by a factor of 0.15.

ii) Since a is negative, the graph is reflected across the x-axis.

iii) Since k is 1925 and h is 150, the graph is shifted up by 1925 units and to the right by 150 units. We will need to adjust the scale to sketch the graph. We will let x range from 0 to 300 and y from – 2000 to 2000.

Ex. 8 A manufacturer has been selling lamps at a price of \$40 per lamp, and at this price, consumers have been buying 5000 lamps per month. The manufacturer wishes to raise the price and estimates that for each \$2 increase in the price, 500 fewer lamps will be sold each month. If it costs the manufacturer \$28 per lamp to produce the lamp, express the monthly profit as a function of the price that the lamps are sold, draw the graph, and estimate the optimal selling price.

Solution:

Let x = price of the lamp

and let y = the number of lamps sold per month

When $x_1 = \$40$, $y_1 = 5000$ lamps. For every \$2 increase in price, 500 fewer lamps are sold. Thus, when $x_2 = \$42$, $y_2 = 4500$ lamps. We can compute the slope of this line:

$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4500 - 5000}{42 - 40} = \frac{-500}{2} = -250$. Using the point-slope formula, we can find the relationship between the number of lamps sold and the price:

$$y - y_1 = m(x - x_1)$$
$$y - 5000 = -250(x - 40)$$
$$y - 5000 = -250x + 10000$$
$$y = -250x + 15000, \text{ Domain: } 0 \le x \le 60.$$

The revenue is equal to price time quantity. So,
$R(x) = xy = x(-250x + 15000) = -250x^2 + 15000x$.

The cost is cost per lamp times the number of lamps. So,
$C(x) = 28y = 28(-250x + 15000) = -7000x + 420000$.
Thus, the profit function is: $P(x) = R(x) - C(x)$
$= -250x^2 + 15000x - (-7000x + 420000)$
$= -250x^2 + 22000x - 420000$
So, $P(x) = -250x^2 + 22000x - 420000$.
Since the profit function is a quadratic equation and the coefficient of the squared term is negative, the graph points down. We can find the x-coordinate of the vertex using $h = -\frac{b}{2a}$:

$$h = \frac{-22000}{2(-250)} = \frac{-22000}{-500} = 44.$$

Thus, $k = P(44) = -250(44)^2 + 22000(44) - 420000$
$= -484000 + 968000 - 420000 = \$64000$.
So, the optimal selling price is \$44, which will yield a maximum profit of \$64,000.

We can use this information to sketch the graph:
The equation in graphing form is $P(x) = -250(x - 44)^2 + 64000$

i) Since $|a| = 250$, the graph is stretched by a factor of 250.
ii) Since a is negative, the graph is reflected across the x-axis.
iii) Since k is 64,000 and h is 44, the graph is shifted up by 64000 units and to the right by 44 units. We will need to adjust the scale to sketch the graph. We will let x range from 0 to 60 and y from – 450,000 to 150,000.