<u>Sect 3.2 - Linear Equations in Two Variables</u>

Concept #1   Solutions to Linear Equations in Two Variables

In this chapter, we will examine linear equations involving two variables. Such equations have an infinite number of solutions. Much in the same way we did with inequalities, we will use a graph to represent the solution to these linear equations. To see how this works, let us begin some definitions.

Definition of a Linear Equation in Two Variables
Let a, b, and c be real numbers where a and b are not both zero. Then an equation that can be written in the form:

$$ax + by = c$$

is called a **linear equation in two variables**.

<u>Find some solutions to the following linear equation:</u>

Ex. 1   x + y = 5
        <u>Solution:</u>
        To find some solutions to this equation, we simply need to find two
        numbers whose sum is five:
                x = 2 and y = 3   or the ordered pair (2, 3)
                x = 1 and y = 4   or the ordered pair (1, 4)
                x = 0 and y = 5   or the ordered pair (0, 5)
                        Etc.
We could keep on generating a list of solutions for this equation until the end of time and we still would not be able to write them all down. We need a better way to represent all the solutions to this equation.

But, suppose we could write down all the possible solutions to a linear equations in two variables. If we then represented those solutions as points on the coordinate system, we would get a straight line. But, we actually cannot write down the all the solutions to the linear equation. So instead, we write down three solutions, plot them as points and use a straight edge to draw a line through those points. That line represents all the solutions to that linear equation. Let us try that with example #1. We will graph the three points we found in the same fashion as we graphed points in the last section. Then, we will take a straight edge and draw a line through those three points:

Sketch the graph of the following equations:

Ex. 2   x + y = 5

Solution:
Recall the three solutions
we find in example one:
(2, 3), (4, 1) and (5, 0).
Let' s plot these as points
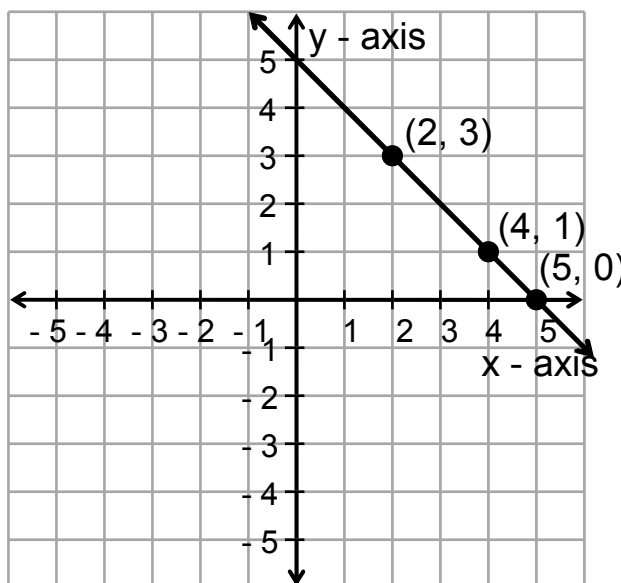on our coordinate system:
(2, 3) - right 2, up 3.
(4, 1) - right 4, up 1.
(5, 0) - right 5.
Now, using a straight edge,
draw a line through the points.
We put arrows on the ends of
the line to indicate that the
line extends on forever in both directions.

There is a couple of observations we want to make. First, every point on the line represents a solution to the equation x + y = 5. So, since the point (0.8, 4.2) lies on the line, then x = 0.8 and y = 4.2 is a solution to our equation. Every point that does not lie on the line is not a solution to the linear equation. Thus, the point (− 3, 2) does not lie on the line so it does not represent a solution to x + y = 5. In fact, if we substitute − 3 for x and 2 for y, we will get a false statement − 3 + 2 ≠ 5.

Determine if the following points are solution to 6x − 5y = 15:

Ex. 3a   (− 5, − 9)   Ex. 3b   (3, − 2)

Ex. 3c $(\frac{5}{3}, -1)$

Solution:
a)   6x − 5y = 15   (replace x by − 5 and y by − 9)
       6(− 5) − 5(− 9) = 15
       − 30 + 45 = 15
       15 = 15   True
So, (− 5, − 9) is a solution the 6x − 5y = 15.

b)   6x − 5y = 15   (replace x by 3 and y by − 2)
       6(3) − 5(− 2) = 15
       18 + 10 = 15
       28 = 15   False
So, (3, − 2) is not a solution the 6x − 5y = 15.

c) $6x - 5y = 15$ (replace x by $\frac{5}{3}$ and y by – 1)

$6(\frac{5}{3}) - 5(-1) = 15$

$10 + 5 = 15$

$15 = 15$ True

So, $(\frac{5}{3}, -1)$ is a solution the $6x - 5y = 15$.

Concept #2   Graphing Linear Equations in Two Variables by Point Plotting

To graph a linear equation in two variables, we will first find three solutions to the equation. We usually organize these in a table. Then we will plot the points on the coordinate system and draw a straight line through the points. Let us try some more examples.

Sketch the graph of the following:

Ex. 4 $3x + 2y = 6$

Solution:
First we need to find three solutions. Let us just pick some numbers for x and find the corresponding y values.

x = 0: $3(0) + 2y = 6$
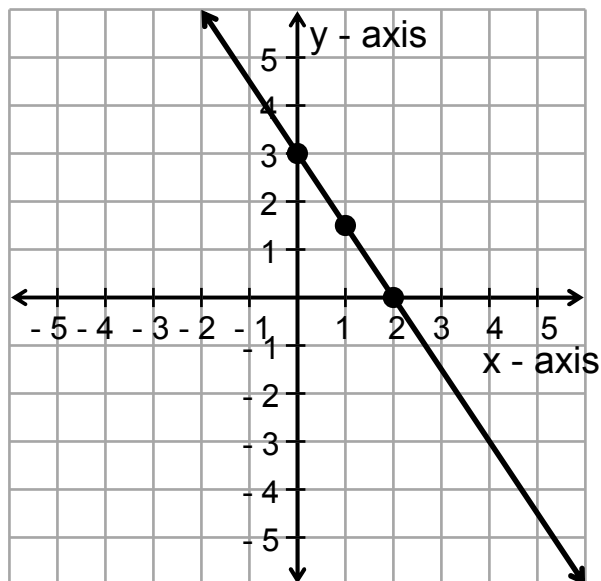$2y = 6$
$y = 3$

x = 1: $3(1) + 2y = 6$
$2y = 3$
$y = 1.5$

x = 2: $3(2) + 2y = 6$
$2y = 0$
$y = 0$

Now organize this in a table and plot the points:

x	y
0	3
1	1.5
2	0

The point where the line intersects the y - axis is called the **y-intercept**. The x-coordinate of the y-intercept is always zero. So, for our equation, (0, 3) is the y-intercept. The **x-intercept** is the point where the line intersects the x-axis. The y-coordinate of the x-intercept is always zero.
In this example, the x-intercept is (2, 0).

Ex. 5 $2x - 5y = 10$

Solution:
Let's use the intercepts as two of our points:
If $y = 0$, then $2x = 10$ or $x = 5$.
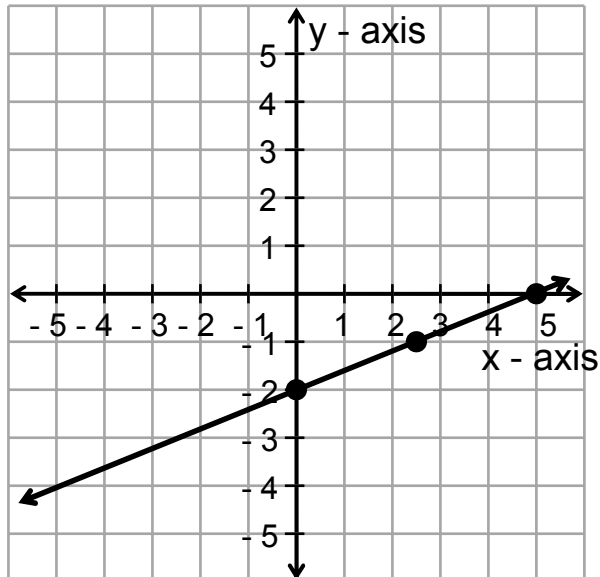If $x = 0$, then $-5y = 10$ or $y = -2$.
For the third point, let $y = -1$
$2x - 5(-1) = 10$
        $2x = 5$ or $x = 2.5$

x	y
5	0
0	-2
2.5	-1

Now, plot the points and graph the line.

Now, let us state the definition of the x- and y-intercepts more formally:

Concept #3   x - and y - Intercepts

Definition
An **x-intercept** is a point (a, 0) where the graph intersects the x-axis.
A **y-intercept** is a point (0, b) where the graph intersects the y-axis.

Finding x- and y-intercepts
Step 1. Find the x-intercept(s) by replacing y by zero and solving for x.
Step 2. Find the y-intercept(s) by replacing x by zero and solving for y.

a) Find the x-intercept, b) the y-intercept, and c) sketch the graph:

Ex. 6 $4x - 3y = 12$

Solution:
a)   To find the x-intercept, replace y by 0 and solve:

$4x - 3(0) = 12$
$4x - 0 = 12$
$4x = 12$

$\dfrac{4x}{4} = \dfrac{12}{4}$

$x = 3$
So, the x-intercept is (3, 0).

b)   To find the y-intercept, replace x by 0 and solve:

$4(0) - 3y = 12$
$0 - 3y = 12$
$-3y = 12$

$\dfrac{-3y}{-3} = \dfrac{12}{-3}$

$y = -4$
So, the y-intercept is (0, -4)

c) Before we graph, let us find a third point:
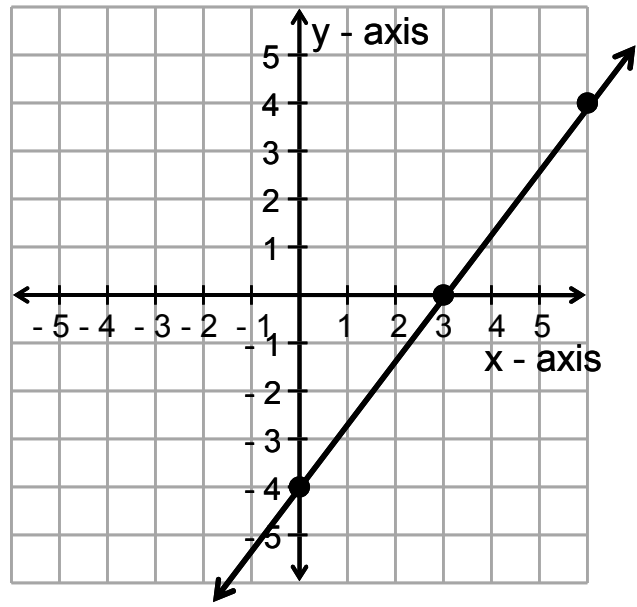Let y = 4:

$$4x - 3(4) = 12$$
$$4x - 12 = 12$$
$$4x = 24$$
$$x = 6$$

Thus, our table looks like:

x	y
3	0
0	- 4
6	4

Now, graph the line.

Ex. 7 $3x - 7y = 7$

Solution:

a)   To find the x-intercept, replace y by 0 and solve:

$$3x - 7(0) = 7$$
$$3x - 0 = 7$$
$$3x = 7$$
$$x = 7/3$$

So, the x-intercept is $(\frac{7}{3}, 0)$.

b)   To find the y-intercept, replace x by 0 and solve:

$$3(0) - 7y = 7$$
$$0 - 7y = 7$$
$$- 7y = 7$$
$$y = - 1$$

So, the y-intercept is $(0, - 1)$

c)   Before we graph, let us find a third point:
Let y = 2:

$$3x - 7(2) = 7$$
$$3x - 14 = 7$$
$$3x = 21$$
$$x = 7$$

Thus, our table looks like:

x	y
$\frac{7}{3}$	0
0	- 1
7	2

Now, graph the line.
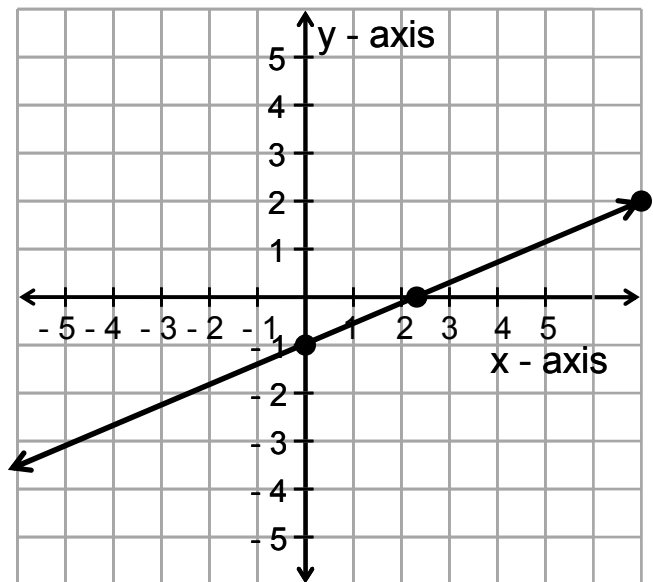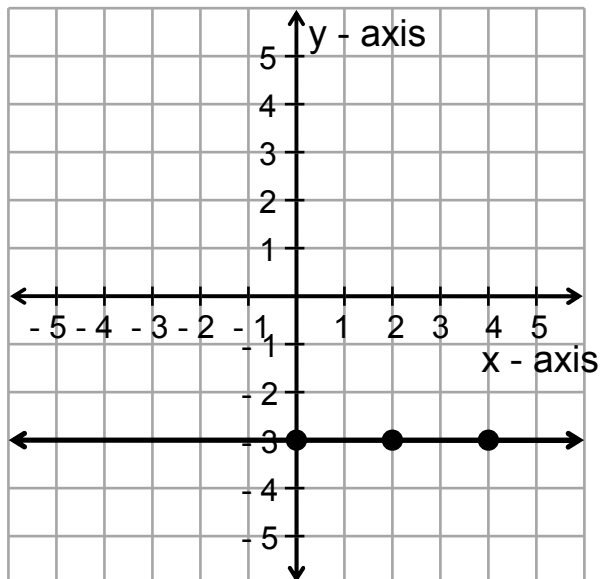
Concept #4   Horizontal and Vertical Lines

Ex. 8 $y = -3$

Solution:
Since there is no variable x
in the equation, we can make
x whatever we want and y will
still be -3:

x	y
0	-3
2	-3
4	-3

Plotting the points and
drawing a line, we get a
horizontal line. Notice the
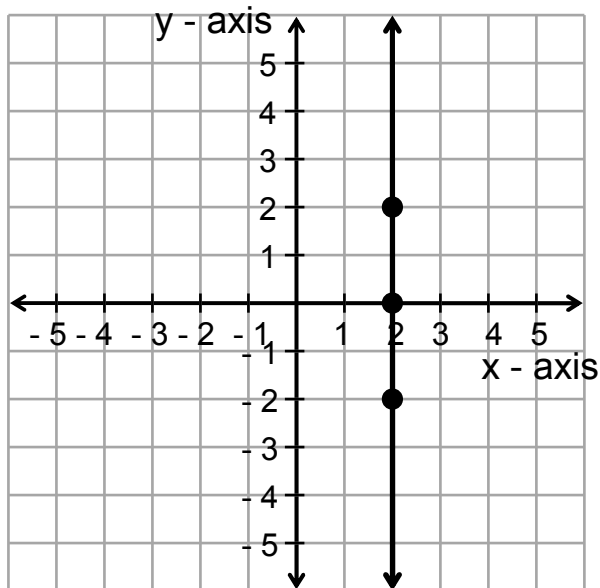y-intercept is $(0, -3)$ and
there is no x-intercept.

Ex. 9 $x = 2$

Solution:
Since there is no variable y
in the equation, we can make
y whatever we want and x will
still be 2:

x	y
2	-2
2	0
2	2

Plotting the points and
drawing a line, we get a
vertical line. Notice the
x-intercept is $(2, 0)$ and
there is no y-intercept.

Definition of Horizontal and Vertical Lines

1.   If the equation is in the form y = constant, then it will be horizontal line.

2.   If the equation is in the form x = constant, then it will be vertical line.

Some hints/techniques for graphing linear equations.

In finding three solutions to a linear equations, we can start with picking a number for either x or y, substituting the number into the equation and solving for the other variable. We typically try to pick integers between – 5 and 5 if we can so we can keep our answers small.

<u>Graph each linear equation:</u>

Ex. 10   x – 3y = 6
   <u>Solution:</u>
   Let's start with the intercepts:
   If x = 0, then – 3y = 6 or y = – 2.
   If y = 0, then x = 6.
   For a third point, try y = – 1.
   If y = – 1, then x – 3(–1) = 6
       x + 3 = 6
       x = 3
   So, our table looks like:

x	y
0	– 2
6	0
3	– 1

   Now, draw the graph.
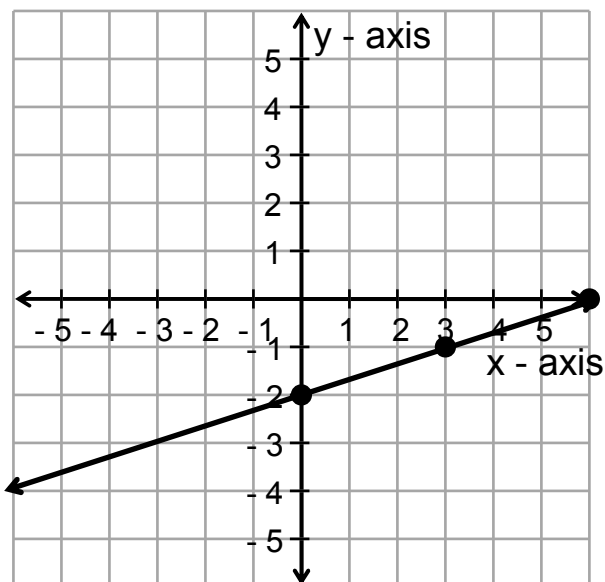
Ex. 11   x = – 2y + 1
   <u>Solution:</u>
   Here, it is easiest to pick values for y. Let us pick 0, 1, and 2 for y:

 | x | y |
   |---|---|
   | – 2(0) + 1 = 1 | 0 |
   | – 2(1) + 1 = – 1 | 1 |
   | – 2(2) + 1 = – 3 | 2 |

   Now, draw the graph.
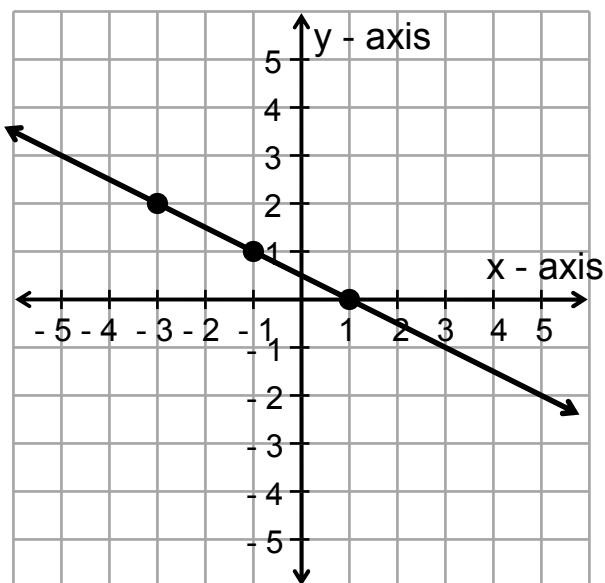
Ex. 12 $y = \frac{2}{3}x + 1$

Solution:
Here, it is easiest to pick values for x. To avoid fractions, we will pick x to be multiplies of three.

x	y
− 3	$\frac{2}{3}(-3) + 1 = -2 + 1 = -1$
0	$\frac{2}{3}(0) + 1 = 0 + 1 = 1$
3	$\frac{2}{3}(3) + 1 = 2 + 1 = 3$

Now, graph the line.
Notice that the y-intercept is the point (0, 1). The y-coordinate matches the constant term in our equation. In general, if the linear equation is solve for y, the constant term will be the y-coordinate of the y-intercept.
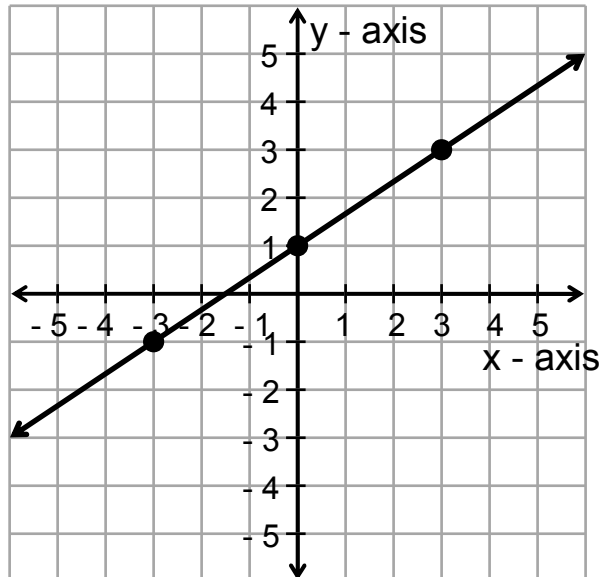
The graph of y = mx + b has a y-intercept of (0, b).

Ex. 13   3y − 10 = 5x

Solution:
Let's choose values for y. To avoid, decimals, let's use multiples of five:
If y = − 5, then
3(− 5) − 10 = − 25 = 5x
         − 5 = x
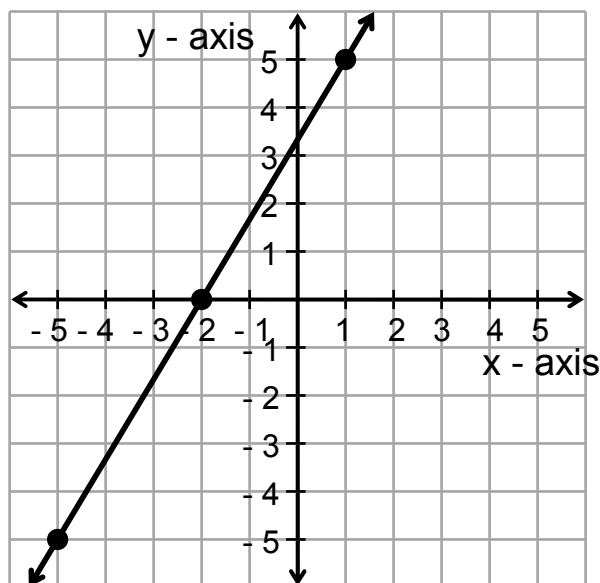If y = 0, then
3(0) − 10 = − 10 = 5x
         − 2 = x
If y = 5, then
3(5) − 10 = 5 = 5x
         1 = x

x	y
− 5	− 5
− 2	0
1	5

Now, graph the line.

Graph each equation on the same set of axes and compare:

Ex. 14 $y = 2x$, $y = 2x - 3$

Solution:

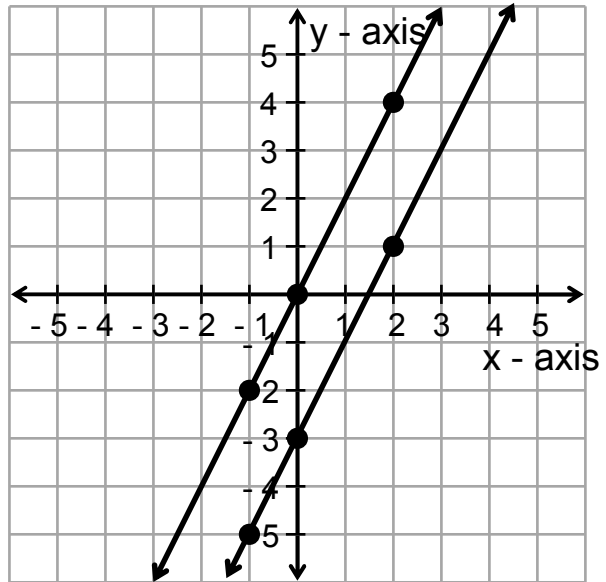It is easiest to pick values for x:

$y = 2x$

x	y
− 1	2(− 1) = − 2
0	2(0) = 0
2	2(2) = 4

$y = 2x - 3$

| x	y
− 1	2(− 1) − 3 = − 5
0	2(0) − 3 = − 3
2	2(2) − 3 = 1

Now, graph both lines.

Notice that the lines are parallel. Also note that the coefficient of the x-term is the same. If the equations are solved for y and the coefficient of the x-term (m) are the same, the lines are parallel.

Ex. 15   The revenue y (in millions of dollars) for a company is given by $y = 4x + 16$, where x is the number of years after 2004. Graph the equation and use the graph to predict the revenue is 2013.

Solution:

We only need to use the first quadrant since time and revenue cannot be negative. We will also have to use a different scale for the y-axis. Letting the maximum value of y be 80 should work. Since there are ten units, then each unit is worth 80/10 = 8 units.
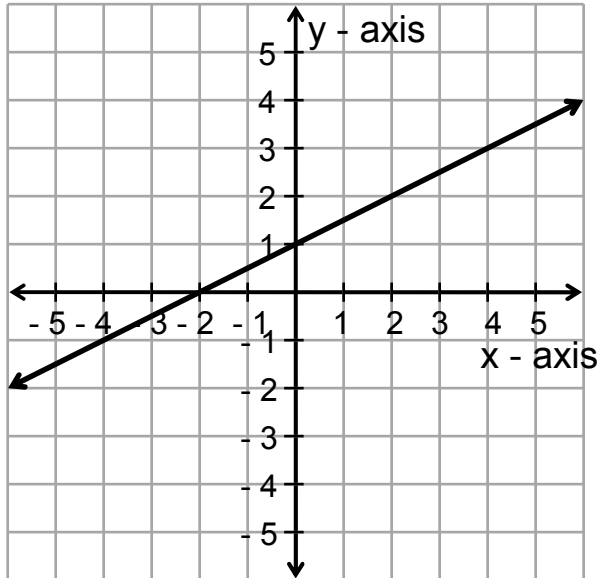
x	y
0	4(0) + 16 = 16
2	4(2) + 16 = 24
4	4(4) + 16 = 32
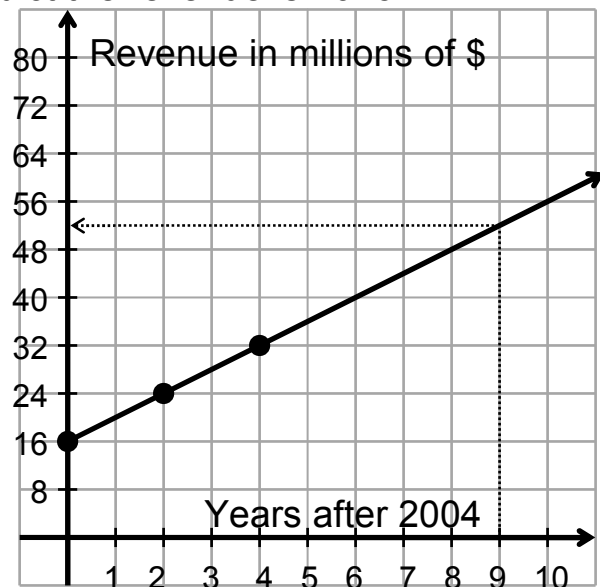
Since 2013 is nine years after 2004, then x = 9. Using the graph, the y - value is halfway between 48 and 56. So, the revenue will be $52 million in 2013.

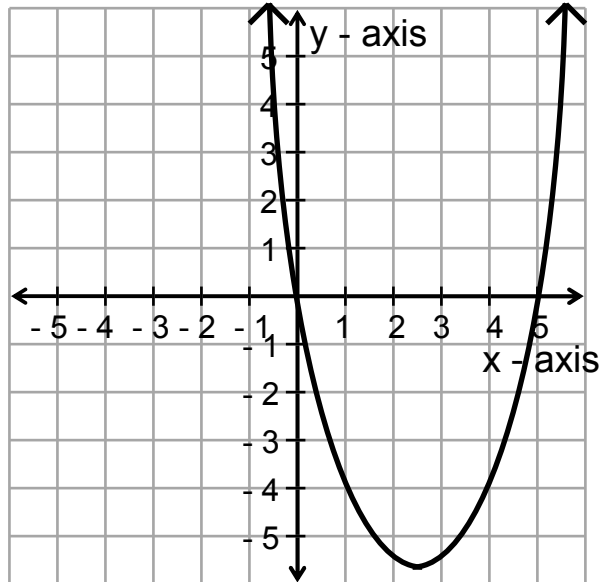**Given the graphs below, find the x-intercept(s) and y-intercept(s):**

Ex . 16

Solution:
x-intercept(s): (– 2, 0)
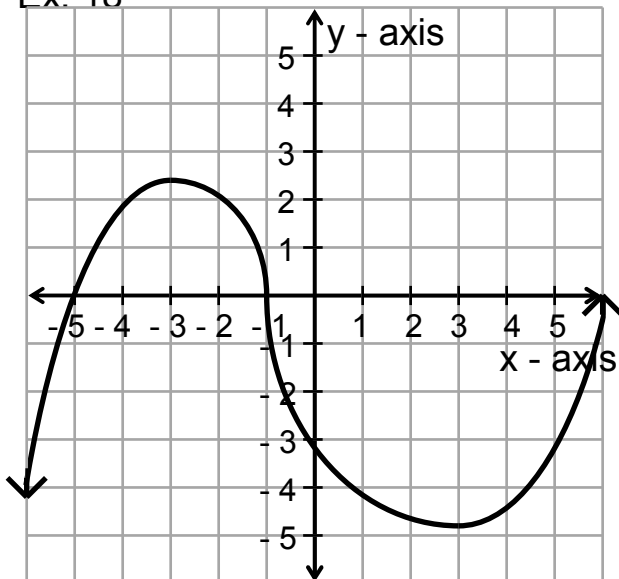y-intercept(s): (0, 1)

Ex. 17

Solution:
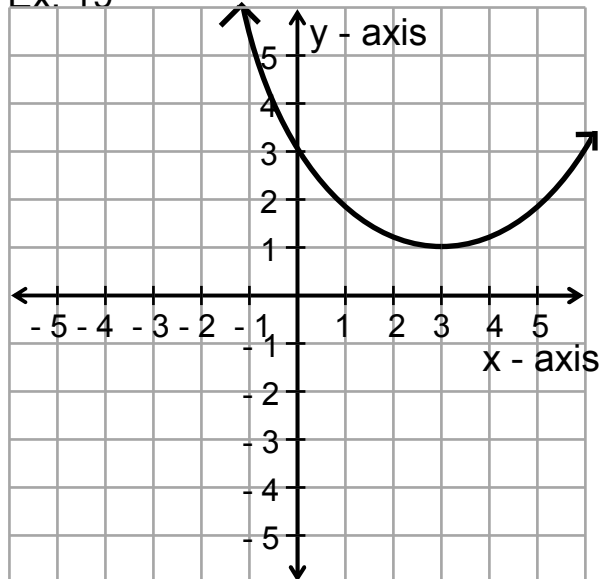x-intercept(s): (0, 0) & (5, 0)
y-Intercept(s): (0, 0)

Ex. 18

Solution:
x-intercept(s): (– 5, 0),
        (– 1, 0), & (6, 0)
y-intercept: (0, – 3)

Ex. 19

Solution:
x-intercept(s): None

y-intercept: (0, 3)