Sect 6.1 - Greatest Common Factor and Factoring by Grouping

Our goal in this chapter is to solve non-linear equations by breaking them down into a series of linear equations that we can solve. To do this, we **factor** the non-linear polynomial. We might be given a problem like this:

Solve: $18x^3 + 45x^2 - 32x - 80 = 0$

We "factor" $18x^3 + 45x^2 - 32x - 80$ by the techniques developed in this chapter to get: $(3x - 4)(3x + 4)(2x + 5) = 0$

If we know that a product is zero, then at least one of the factors has to be zero. So, we can break the problem into:

$(3x - 4) = 0$	$(3x + 4) = 0$	and	$(2x + 5) = 0$ and solve
$3x = 4$	$3x = -4$		$2x = -5$
$x = \frac{4}{3}$	$x = -\frac{4}{3}$	and	$x = -\frac{5}{2}$

Thus, this problem has three solutions: $\left\{-\frac{5}{2}, -\frac{4}{3}, \frac{4}{3}\right\}$

Factoring is the reverse of what we did in chapter five; it helps us write a sum as a product which in turn, will allow us to solve equations like the one above.

Concepts #1 - #3: Factoring out the G.C.F.

The **Greatest Common Factor** (G.C.F.) is the largest factor that can divide into the terms of an expression evenly with no remainder. When we factor out the G.C.F., we are using the distributive property backwards to "undistribute" the G.C.F. For instance, $12x - 12y$ has a common factor of 12, so we can rewrite $12x - 12y$ as $12(x - y)$.

Factor out the G.C.F.

Ex. 1a) $9x - 9y$

Ex. 1b) $7a - 5ax$

Ex. 1c) $-27x + 27$

Ex. 1d) $8x - 16$

Ex. 1e) $-36x - 24y$

Solution:

a) The G.C.F. = 9, so $9x - 9y = 9(x - y)$.
b) The G.C.F. = a, so $7a - 5ax = a(7 - 5x)$.
c) Since the first term is negative, we will factor out a negative with the G.C.F. The G.C.F. = – 27, so $-27x + 27 = -27(x - 1)$.
d) The G.C.F. = 8, so $8x - 16 = 8(x - 2)$.
e) The G.C.F. = – 12, so $-36x - 24y = -12(3x + 2y)$.

To see how to find the G.C.F. of variables with powers, let's take a closer look at example #1 d and e. Let's write 8 as 2^3, 16 as 2^4, 36 as $2^2 \bullet 3^2$, 24 as $2^3 \bullet 3$, and 12 as $2^2 \bullet 3$. So, the problems then would look like:

$8x - 16 = 8(x - 2)$

$2^3x - 2^4 = 2^3(x - 2)$

Notice that the two with the lower power was the G.C.F.

$-36x - 24y = -12(3x + 2y)$

$-2^2 \bullet 3^2x - 2^3 \bullet 3y = -2^2 \bullet 3(3x + 2y)$

Notice that the factors of 2 and 3 with the lowest powers were used for the G.C.F.

So, to find the G.C.F. of variables, list the variables that appear in all the terms and then choose the lowest power of each respective variables as the powers for the G.C.F.

Factor out the G.C.F.

Ex. 2a $x^2 - xy$

Solution:

Since x is the only variable and the lowest power is one, then the G.C.F. = x. Thus,

$x^2 - xy = x(x - y)$

Ex. 2b $x^3y^6 - x^4y^5$

Solution:

Both terms have x and y in common. The lowest power of x is 3 and of y is 5, so the G.C.F. = x^3y^5. Thus,

$x^3y^6 - x^4y^5 = x^3y^5(y - x)$.

Ex. 3 $8x^2y^3 - 12xy^4$

Solution:

The G.C.F. of 8 and 12 is four. Both terms have x and y in common. The lowest power of x is 1 and of y is 3. So, the G.C.F. = $4xy^3$. Thus,

$8x^2y^3 - 12xy^4 = 4xy^3(2x - 3y)$

Ex. 4 $-9xy^2 + 27x^2y$

Solution:

The G.C.F. = $-9xy$. So, $-9xy^2 + 27x^2y = -9xy(y - 3x)$

Ex. 5 $3axy^2 - 2ax^2yz + ax^2y^2z^3$

Solution:

The G.C.F. = axy. Notice we did not include z since there is no z in the first term. Thus,

$3axy^2 - 2ax^2yz + ax^2y^2z^3 = axy(3y - 2xz + xyz^3)$

Ex. 6 $7x^3y^4 - 14x^4y^4 + 21x^4y^3$

Solution:

The G.C.F = $7x^3y^3$. So, $7x^3y^4 - 14x^4y^4 + 21x^4y^3 = 7x^3y^3(y - 2xy + 3x)$

Concept #4 Factoring out a Binomial Factor

Ex. 7 $3x(2x - 5) + 4(2x - 5)$

Solution:

The binomial $(2x - 5)$ is the G.C.F., so factor it out the same way.

$3x\mathbf{(2x - 5)} + 4\mathbf{(2x - 5)} = \mathbf{(2x - 5)}(3x + 4)$

Ex. 8 $12x(4x - 3) - 16y(4x - 3)$

Solution:

The G.C.F. = $4(4x - 3)$, so $12x\mathbf{(4x - 3)} - 16y\mathbf{(4x - 3)}$

$= 4\mathbf{(4x - 3)}(3x - 4y)$.

Ex. 9 $6x(2x + 7) - (2x + 7)^2$

Solution:

The G.C.F. = $(2x + 7)$. Thus, $6x(2x + 7) - (2x + 7)^2$

$= (2x + 7)(6x - (2x + 7))$ Now, simplify inside the parenthesis.

$= (2x + 7)(6x - 2x - 7) = (2x + 7)(4x - 7)$

Ex. 10 $-21(x - 3)^3(y + 2)^2 + 28(x - 3)^2(y + 2)^3$

Solution:

G.C.F. = $-7(x - 3)^2(y + 2)^2$, so

$-21\mathbf{(x - 3)}^3(y + 2)^2 + 28\mathbf{(x - 3)}^2(y + 2)^3$

$= -7\mathbf{(x - 3)}^2(y + 2)^2[3\mathbf{(x - 3)} - 4(y + 2)]$

Now, simplify inside the brackets.

$= -7(x - 3)^2(y + 2)^2[3x - 9 - 4y - 8]$

$= -7(x - 3)^2(y + 2)^2[3x - 4y - 17]$

Concept #5 Factoring by Grouping

Whenever a factoring problem has four or more terms, we put the terms into groups and factor out the G.C.F. of each group. We then see if the "groupings" have any pieces in common. If so, we then factor out the common pieces. Let's look at some examples.

<u>Factor the following by grouping:</u>

Ex. 11 $my + 6y - 5m - 30$

Solution:

Let's put the first two terms in one grouping and the last two terms in the other grouping:

$my + 6y - 5m - 30 = my + 6y + -5m - 30 = (my + 6y) + (-5m - 30)$

The G.C.F. of the first grouping is y and G.C.F. of the second

grouping is – 5, so
(my + 6y)+ (– 5m – 30) = y(m + 6) – 5(m + 6)
Now, the two pieces have (m + 6) in common, so we will factor that out: y(m + 6) – 5(m + 6) = (m + 6)(y – 5)

Always be sure to factor out the G.C.F. of all the terms before factoring by grouping.

Ex. 12 $10uv^3 – 5v^3 – 30uv^2 + 15v^2$

<u>Solution:</u>
The G.C.F. = $5v^2$, so $10uv^3 – 5v^3 – 30uv^2 + 15v^2$
$= 5v^2[2uv – v – 6u + 3]$ Now, let's put the first two terms in one grouping and the last two terms in the other grouping:
$5v^2[2uv – v – 6u + 3] = 5v^2[(2uv – v) + (– 6u + 3)]$
The G.C.F. of the first grouping is v and G.C.F. of the second grouping is – 3, so
$5v^2[(2uv – v) + (– 6u + 3)] = 5v^2[v(2u – 1) – 3(2u – 1)]$
Now, the two pieces have (2u – 1) in common, so we will factor that out: $5v^2[v(2u – 1) – 3(2u – 1)] = 5v^2(2u – 1)(v – 3)$

Ex. 13 $w^2 + 20z + 5w + 4wz$

<u>Solution:</u>
The G.C.F. = 1, so let's put the first two terms in one grouping and the last two terms in the other grouping and factor out the G.C.F. in each group:
$w^2 + 20z + 5w + 4wz = (w^2 + 20z) + (5w + 4wz)$
$= 1(w^2 + 20z) + w(5 + 4z)$ Here, we are stuck. When this happens, you want to try a different arrangement. Perhaps group 5w with w^2 instead: $w^2 + 20z + 5w + 4wz = w^2 + 5w + 20z + 4wz$
$= (w^2 + 5w) + (20z + 4wz) = w(w + 5) + 4z(5 + w)$ (switch w & 5)
$= w(w + 5) + 4z(w + 5) = (w + 5)(w + 4z)$

Ex. 14 $5p^3 – 10p^2 + 15p + 33$

<u>Solution:</u>
The G.C.F. = 1, so let's put the first two terms in one grouping and the last two terms in the other grouping and factor out the G.C.F. in each group:
$5p^3 – 10p^2 + 15p + 33 = (5p^3 – 10p^2) + (15p + 33)$
$= 5p^2(p – 2) + 3(5p + 11)$ Here, we are stuck. So, let's try grouping $5p^3$ and 15p:

$5p^3 - 10p^2 + 15p + 33 = 5p^3 + 15p - 10p^2 + 33$
$= (5p^3 + 15p) + (-10p^2 + 33) = 5p(p^2 + 3) - (10p^2 - 33)$ No, this does not work either. Our last possibility is to pair $5p^3$ with 33:
$5p^3 - 10p^2 + 15p + 33 = 5p^3 + 33 - 10p^2 + 15p$
$= (5p^3 + 33) + (-10p^2 + 15p) = 1(5p^3 + 33) - 5p(2p - 3)$ This does not work either. This means that $5p^3 - 10p^2 + 15p + 33$ cannot be factored. If a problem cannot be factored, then we say that it is **prime**. So, $5p^3 - 10p^2 + 15p + 33$ is prime.

Ex. 15 $8ax^5y^2 - 24bx^5y^2 - a + 3b$

Solution:

The G.C.F. = 1, so let's put the first two terms in one grouping and the last two terms in the other grouping and factor out the G.C.F. in each group:
$8ax^5y^2 - 24bx^5y^2 - a + 3b = (8ax^5y^2 - 24bx^5y^2) + (-a + 3b)$
$= 8x^5y^2(a - 3b) - (a - 3b) = (a - 3b)(8x^5y^2 - 1)$