Sect 6.7 - Solving Equations Using the Zero Product Rule

Concept #1: Definition of a Quadratic Equation

A **quadratic equation** is an equation that can be written in the form $ax^2 + bx + c = 0$ (referred to as standard form) where a, b, and c are real numbers and $a \neq 0$.

A quadratic equation can be thought of a equation involving a second degree polynomial.

Are the following quadratic equations?

Ex. 1a $3x + 11 = 8$ Ex. 1b $9y^2 - 6y + 2 = 0$

Ex. 1c $\frac{4}{13}w^3 - w^2 = 52$ Ex. 1d $8x^2 - 27 = 37$

Solution:

a) The power of x is one, so this is a not a quadratic equation. **No**
b) The highest power of y is 2, so this is a quadratic equation. **Yes**
c) The highest power of w is 3, so this is not a quadratic equation. **No**
d) The highest power of x is 2, so this is a quadratic equation. **Yes**

Concept #2 Zero Product Rule

If the product of two numbers is zero, then at least one of the numbers has to be zero. This is known as the **Zero Product Rule**.

Zero Product Rule

If a and b are real numbers and if $a \bullet b = 0$, then either $a = 0$ or $b = 0$.

When we are solving equations, we are trying to find all possible numbers that make the equation true. If we have a product of factors equal to zero, we will use the Zero Product Rule to set each factor equal to zero. Then, we will solve each resulting equation. This will give us a list of all possible real solutions to the original equation.

Concept #3 Solving Equations Using the Zero Product Rule

It is important to follow these three steps when solving equations by factoring:

1) Be sure that you have 0 on one side of the equation.
2) Be sure that you have a product on the other side of the equation. If not, use the techniques established in this chapter to factor the other side of the equation into a product.
3) Once steps #1 and #2 are satisfied, use the zero product rule to set each factor equal to zero, and solve.

Solve the following:

Ex. 2 $3x(2x - 5)(x + 4) = 0$

Solution:

Set each factor equal to zero and solve:

$3x(2x - 5)(x + 4) = 0$ (Zero Product Rule)

$3x = 0$	or	$(2x - 5) = 0$	or	$(x + 4) = 0$
$\frac{3x}{3} = \frac{0}{3}$	or	$2x - 5 = 0$	or	$x + 4 = 0$
		$\underline{\quad + 5 = + 5}$		$\underline{\quad - 4 = - 4}$
$x = 0$	or	$\frac{2x}{2} = \frac{5}{2}$	or	$x = - 4$
$x = 0$	or	$x = 2.5$	or	$x = - 4$

So, our solution is x can be – 4, 0, or 2.5. We will write this solution as the set {– 4, 0, 2.5}.

We always like to list the numbers in the set from the smallest to the largest. If the problem happens to have no solution, we will write the empty set { }. Some books will also write $\varnothing$ for the empty set, but we stick with "no solution" or { }.

Ex. 3 $6(2x - 7)(3x - 5) = 0$

Solution:

Set each factor equal to zero and solve:

$6(2x - 7)(3x - 5) = 0$ (Zero Product Rule)

$6 = 0$	or	$(2x - 7) = 0$	or	$(3x - 5) = 0$
No Solution	or	$2x - 7 = 0$	or	$3x - 5 = 0$
		$\underline{\quad + 7 = + 7}$		$\underline{\quad + 5 = + 5}$
	or	$\frac{2x}{2} = \frac{7}{2}$	or	$\frac{3x}{3} = \frac{5}{3}$
	or	$x = 3.5$	or	$x = 5/3$

So, our answer is $\{\frac{5}{3}, 3.5\}$.

Ex. 4 $18x^3 + 45x^2 - 32x - 80 = 0$

Solution:

The G.C.F. = 1. Since there are four terms, we need to factor by grouping. The first two terms have $9x^2$ in common and the last two terms have – 16 in common.

$18x^3 + 45x^2 - 32x - 80 = 0$ (group)

$(18x^3 + 45x^2) + (-32x - 80) = 0$ (factor out $9x^2$ and – 16)

$9x^2(2x + 5) - 16(2x + 5) = 0$ (factor out $(2x + 5)$)

$(2x + 5)(9x^2 - 16) = 0$

If we look at the 2nd factor, $9x^2 - 16$, it is a difference of squares:

$(2x + 5)(9x^2 - 16) = 0$ ($F = 3x$ and $L = 4$)

Since $F^2 - L^2 = (F - L)(F + L)$, then

$(2x + 5)(9x^2 - 16) = 0$

$(2x + 5)(3x - 4)(3x + 4) = 0$ (Zero Product Rule)

$2x + 5 = 0$ or $3x - 4 = 0$ or $3x + 4 = 0$ (solve)

$2x = -5$ or $3x = 4$ or $3x = -4$

$x = -\frac{5}{2}$ or $x = \frac{4}{3}$ or $x = -\frac{4}{3}$

So, our answer is $\{-\frac{5}{2}, -\frac{4}{3}, \frac{4}{3}\}$.

Ex. 5 $256x^4 = -500x$

Solution:

First, get zero on one side by adding 500x to both sides:

$$\begin{array}{r} 256x^4 = -500x \\ \underline{+500x = +500x} \\ 256x^4 + 500x = 0 \end{array}$$

G.C.F. = 4x, so $256x^4 + 500x = 4x(64x^3 + 125) = 0$

But, $64x^3 + 125$ is a sum of cubes with $F = 4x$ and $L = 5$. Since $F^3 + L^3 = (F + L)(F^2 - FL + L^2)$, then

$4x(64x^3 + 125) = 0$

$4x(4x + 5)((4x)^2 - (4x)(5) + (5)^2) = 0$

$4x(4x + 5)(16x^2 - 20x + 25) = 0$ (Zero Product Rule)

$4x = 0$ or $4x + 5 = 0$ or $16x^2 - 20x + 25 = 0$

$x = 0$ or $x = -1.25$ or No real solution*

So, our answer is $\{-1.25, 0\}$.

*– If the resulting trinomial from using the sum or difference of cubes is degree two, it is not factorable and hence will not contribute any real solutions. It does have two *complex solutions* which we will study in the next course.

Ex. 6 $12x^2 - x = 6$

<u>Solution:</u>

First, get zero on one side by subtracting 6 from both sides:

$$\begin{array}{r} 12x^2 - x = 6 \\ \underline{\quad - 6 = -6} \\ 12x^2 - x - 6 = 0 \end{array} \qquad \text{(trial and error)}$$

$x \bullet 12x$	$-1 \bullet 6$
$2x \bullet 6x$	$-2 \bullet 3$
$3x \bullet 4x$	

$(3x + 3)(4x - 2)$ — No, G.C.F. $\neq 1$

$(3x - 2)(4x + 3)$

$$\begin{array}{ll} \text{O.} & 9x \\ \underline{\text{I.}} & \underline{-8x} \\ & x \end{array}$$

Yes, change signs.

$(3x + 2)(4x - 3) = 0$ (Zero Product Rule)

$3x + 2 = 0$ or $4x - 3 = 0$

$x = -\frac{2}{3}$ or $x = \frac{3}{4}$

So, our answer is $\left\{-\frac{2}{3}, \frac{3}{4}\right\}$.

Ex. 7 $32x^5 + 243 = 36x^2(2x + 3)$

<u>Solution:</u>

First, get zero on one side by subtracting $36x^2(2x + 3)$ from both sides:

$$\begin{array}{r} 32x^5 + 243 = 36x^2(2x + 3) \\ \underline{-36x^2(2x + 3) = -36x^2(2x + 3)} \end{array}$$

$32x^5 + 243 - 36x^2(2x + 3) = 0$ (distribute)

$32x^5 + 243 - 72x^3 - 108x^2 = 0$ (reorder)

$32x^5 - 72x^3 - 108x^2 + 243 = 0$

The G.C.F. = 1. Since there are four terms, we need to factor by grouping. The first two terms have $8x^3$ in common and the last two terms have -27 in common.

$32x^5 - 72x^3 - 108x^2 + 243 = 0$ (group)

$(32x^5 - 72x^3) + (-108x^2 + 243) = 0$ (factor out $8x^3$ & -27)

$8x^3(4x^2 - 9) - 27(4x^2 - 9) = 0$ (factor out $(4x^2 - 9)$)

$(4x^2 - 9)(8x^3 - 27) = 0$

The first factor is a difference of squares with F = 2x and L = 3. The second factor is a difference of cubes with F = 2x and L = 3. Since $F^2 - L^2 = (F - L)(F + L)$, then

$\mathbf{(4x^2 - 9)}(8x^3 - 27) = 0$

$\mathbf{(2x - 3)(2x + 3)}(8x^3 - 27) = 0$

Since $F^3 - L^3 = (F - L)(F^2 + FL + L^2)$, then

$(2x - 3)(2x + 3)\mathbf{(8x^3 - 27)} = 0$

$(2x - 3)(2x + 3)\mathbf{(2x - 3)((2x)^2 + (2x)(3) + (3)^2)} = 0$

$(2x - 3)(2x + 3)\mathbf{(2x - 3)(4x^2 + 6x + 9)} = 0$

Now, use the Zero Product Rule and solve.

$2x - 3 = 0$	$2x + 3 = 0$	$2x - 3 = 0$	$4x^2 + 6x + 9 = 0$
$x = 1.5$	$x = -1.5$	$x = 1.5$	No real solution

Since 1.5 repeats, list it only once. So, our answer is {– 1.5, 1.5} or { ± 1.5}.

The notation $x = \pm a$ means that $x = a$ or $-a$. It is a more compact way of writing the answer. We will see this again in the next course.

Ex. 8 $x^2 = -36$

Solution:

First, get zero on one side by adding 36 to both sides:

$$\begin{array}{r} x^2 = -36 \\ \underline{+36 = +36} \\ x^2 + 36 = 0 \end{array}$$

But, the sum of squares is not factorable in the real numbers, so $x^2 + 36 = 0$ has no real solution. Thus, our answer is { }.

Ex. 9 $h(4h + 19) = 5$

Solution:

It would be incorrect to set $h = 5$ and $4h + 19 = 5$ since if the product of two numbers is five, there is no guarantee that one or the other has to be 5 (in fact, the factors could be 2 and 2.5). This property only works with zero.

First, get zero on one side by subtracting 5 from both sides:

$$\begin{array}{r} h(4h + 19) = 5 \\ \underline{-5 = -5} \\ h(4h + 19) - 5 = 0 \end{array}$$ (distribute)

$4h^2 + 19h - 5 = 0$(trial and error)

$h \bullet 4h$ $\quad -1 \bullet 5$

$2h \bullet 2h$

$(2h - 1)(2h + 5)$		$(h - 1)(4h + 5)$	
O.	$10h$	O.	$5h$
I.	$-2h$	I.	$-4h$
	$8h$ No		h No

$(h + 5)(4h - 1)$
O. $\quad - h$
I. $\quad \underline{20h}$
$\quad 19h$ Yes
$(h + 5)(4h - 1) = 0$ (Zero Product Rule)
$h + 5 = 0 \qquad 4h - 1 = 0$
$h = -5 \qquad h = 0.25$
So, our answer is $\{-5, 0.25\}$.

Ex. 10 $\qquad (2x + 3)(x - 4) = -8(x - 1)$

Solution:

First, get zero on one side by adding $8(x - 1)$ to both sides:

$$(2x + 3)(x - 4) = -8(x - 1)$$
$$\underline{\quad + 8(x - 1) = + 8(x - 1)}$$
$$(2x + 3)(x - 4) + 8(x - 1) = 0$$

It would be incorrect to set $2x + 3$, $x - 4$, 8, and $x - 1$ equal to zero because the left side is not a product. There is addition separating $(2x + 3)(x - 4)$ and $8(x - 1)$. We will need to simplify:

$(2x + 3)(x - 4) + 8(x - 1) = 0$ (F.O.I.L. and distribute)
$2x^2 - 8x + 3x - 12 + 8x - 8 = 0$ (combine like terms)
$2x^2 + 3x - 20 = 0$

Now, factor the left side.

$2x^2 + 3x - 20 = 0$ (trial and error)

$x \bullet 2x \qquad -1 \bullet 20 \qquad (x - 4)(2x + 5)$
$\qquad -2 \bullet 10 \qquad$ O. $\quad 5x$
$\qquad -4 \bullet 5 \qquad$ I. $\quad \underline{-8x}$
$\qquad -3x$ Yes, change the signs

$(x + 4)(2x - 5) = 0$ (Zero Product Rule)
$x + 4 = 0 \qquad 2x - 5 = 0$
$x = -4 \qquad x = 2.5$
So, our answer is $\{-4, 2.5\}$.

Concept #4 Solving Applications by Factoring

We will use a lot of the same techniques for solving applications from previous chapters in this section. The only difference in this section is that we will not usually get a linear equation to solve.

<u>Set-up the equation and solve the following:</u>

Ex. 11 Find two consecutive even integers such that three times the first integer squared minus twice the second integer squared is 76.

<u>Solution:</u>

Let F = the first even integer.

Then (F + 2) = the second consecutive even integer.

Their squares are F^2 and $(F + 2)^2$ respectively, so three times the first squared is $3F^2$ and twice the second squared is $2(F + 2)^2$.

Hence, $3F^2$ minus $2(F + 2)^2$ is 76 becomes:

$3F^2 – 2(F + 2)^2 = 76$ (expand)

$3F^2 – 2(F^2 + 4F + 4) = 76$ (distribute)

$3F^2 – 2F^2 – 8F – 8 = 76$ (combine like terms)

$F^2 – 8F – 8 = 76$ (subtract 76 from both sides)

$F^2 – 8F – 84 = 0$ (trial and error)

$F \bullet F$

$– 1 \bullet 84$

$– 2 \bullet 42$

$– 3 \bullet 28$

$– 4 \bullet 21$

$– 6 \bullet 14$

$– 7 \bullet 12$

Since the coefficient of the squared term is one, we can simply sum the factors of the last terms to find the correct coefficient of the middle term.

$(F – 6)(F + 14)$

O. $14F$

<u>I. $– 6F$</u>

$8F$ Yes, change signs

$(F + 6)(F – 14) = 0$ (Zero Product Rule)

$F + 6 = 0$ or $F – 14 = 0$

$F = – 6$ or $F = 14$

$F + 2 = – 4$ $F + 2 = 16$

So, the integers are either – 6 and – 4 or 14 and 16.

If it had stated in the problem that the integers were positive, then the answer would are been 14 and 16. There are some applications where the answer has to be positive. For example, if we are solving for time or the length of a rectangle, those result have to be positive.

Ex. 12 The width of a rectangle is four meters less than three times the length. If the area of the rectangle is 160 m^2, find the dimensions.

<u>Solution:</u>

Let L = the length of the rectangle

Then (3L – 4) = the width of a rectangle.

The area of a rectangle is $A = L \bullet w$, but $A = 160$ m^2 and $w = 3L – 4$:

$160 = L \bullet (3L – 4)$ (distribute)

$160 = 3L^2 – 4L$ (subtract 160 from both sides)

$3L^2 - 4L - 160 = 0$ (trial and error)

$L \bullet 3L$ $\quad -1 \bullet 160$
$-2 \bullet 80$
$-4 \bullet 40$
$-5 \bullet 32$
$-8 \bullet 20$
$-10 \bullet 16$

$(L - 10)(3L + 16)$
O. $16L$
I. $-14L$
$2L$ No

$(L + 16)(3L - 10)$
O. $-10L$
I. $48L$
$38L$ No

$(L - 8)(3L + 20)$
O. $20L$
I. $-24L$
$-4L$ Yes

$(L - 8)(3L + 20) = 0$ (Zero Product Rule)
$L - 8 = 0$ or $3L + 20 = 0$
$L = 8$ or $L = -20/3$
$3L - 4 = 20$ reject, the length cannot be negative.

The rectangle is 8 ft by 20 ft.

Ex. 13 A ball is thrown from the top of a tower with an upward velocity of 112 feet per second. How many seconds after the ball is thrown will it hit the ground if the ground is 704 feet below the top of the tower. Use the formula $h(t) = vt - 16t^2$ where h(t) is the height above the tower after t seconds, and v is the initial upward velocity.

Solution:

Since h(t) is the height above the tower and the ball lands 704 feet below the top of the tower, then the height will be negative.
Replace v by 112 and h(t) by – 704 and solve:

$h(t) = vt - 16t^2$
$-704 = 112t - 16t^2$ (add 704 to both sides)
$0 = 112t - 16t^2 + 704$ (reorder)
$-16t^2 + 112t + 704 = 0$ (factor out the G.C.F. of – 16)
$-16(t^2 - 7t - 44) = 0$ (trial and error)

Since the coefficient of the squared term is 1, $-11 \bullet 4 = -44$, and $-11 + 4 = -7$, then $t^2 - 7t - 44 = (t - 11)(t + 4)$. So,

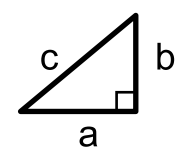
$-16(t^2 - 7t - 44) = 0$ becomes
$-16(t - 11)(t + 4) = 0$ (Zero Product Rule)
$-16 = 0$ or $t - 11 = 0$ or $t + 4 = 0$
No Solution $\quad t = 11 \quad t = -4$, not possible

So, it will take eleven seconds for the ball to hit the ground.

Concept #5 Right Triangles and the Pythagorean Theorem

In a right triangle, there is a special relationship between the length of the legs (a and b) and the hypotenuse (c). This is known as the Pythagorean Theorem:

<u>Pythagorean Theorem</u>



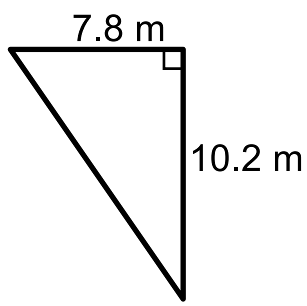
In a right triangle, the square of the hypotenuse (c^2) is equal to the sum of the squares of the legs ($a^2 + b^2$)

$$c^2 = a^2 + b^2$$

Keep in mind that the hypotenuse is the longest side of the right triangle.

<u>Find the length of the missing sides (to the nearest hundredth):</u>

Ex. 14



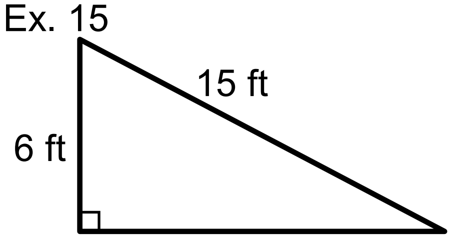
<u>Solution:</u>
In this problem, we have the two legs of the triangle and we are looking for the hypotenuse:

$$c^2 = a^2 + b^2$$
$$c^2 = (7.8)^2 + (10.2)^2$$
$$c^2 = 60.84 + 104.04$$
$$c^2 = 164.88$$

To find c, take the square root* of 164.88:

$$c = \pm\sqrt{164.88} = \pm 12.8405...$$
$$c \approx 12.84 \text{ m}$$

Ex. 15



<u>Solution:</u>
In this problem, we have one leg and the hypotenuse and we are looking for the other leg:

$$c^2 = a^2 + b^2$$
$$(15)^2 = (6)^2 + b^2$$
$$225 = 36 + b^2 \quad \text{(solve for } b^2\text{)}$$
$$\underline{-36 = -36}$$
$$189 = b^2$$

To find b, take the square root of 189:

$$b = \pm\sqrt{189} = \pm 13.7477...$$
$$b \approx 13.75 \text{ ft}$$

* - The equation $c^2 = 164.88$ actually has two solutions ≈ 12.84 and ≈ -12.84, but the lengths of triangles are positive so we ignore the negative solution.

Solve the following:

Ex. 16 The length of the longer leg of a right triangle is 2 ft more than twice the shorter leg. The hypotenuse is 2 ft less than three times the length of the shorter leg. Find the dimensions of the triangle.

Solution:

Let s = the length of the shorter leg

Then 2s + 2 = the length of the longer leg

and 3s – 2 = the length of the hypotenuse.

Recall the Pythagorean Theorem for a right triangle: $c^2 = a^2 + b^2$ where a and b are the lengths of the legs and c is the length of the hypotenuse. Replace a by s, b by (2s + 2) and c by (3s – 2):

$$c^2 = a^2 + b^2$$

$$(3s - 2)^2 = s^2 + (2s + 2)^2 \qquad \text{(expand)}$$

$$9s^2 - 6s - 6s + 4 = s^2 + 4s^2 + 4s + 4s + 4 \quad \text{(combine like terms)}$$

$$9s^2 - 12s + 4 = 5s^2 + 8s + 4 \qquad \text{(subtract } 5s^2 + 8s + 4 \text{ from}$$

$$\underline{-5s^2 - 8s - 4 = -5s^2 - 8s - 4} \qquad \text{both sides)}$$

$$4s^2 - 20s = 0 \qquad \text{(factor out the G.C.F. of 4s)}$$

$$4s(s - 5) = 0 \qquad \text{(Zero Product Rule)}$$

$4s = 0$	or	$s - 5 = 0$
$s = 0$		$s = 5$
Not possible		$2s + 2 = 12$
		$3s - 2 = 13$

The dimensions are 5 ft by 12 ft by 13 ft.

Ex. 17 In a trapezoid, the length of the longer base is three feet more than twice the length of the shorter base. The height is seven feet more than the length of the shorter base. If the area is 108 ft^2, find the length of the shorter base.

Solution:

Let s = the length of the shorter base

Then 2s + 3 = the length of the longer base

and s + 7 = the height

Recall the formula for the area of a trapezoid:

$$A = \frac{1}{2}(b_1 + b_2)h$$

Replacing A by 108, b_1 by s, b_2 by 2s + 3, and h by s + 7, we get:

$$A = \frac{1}{2}(b_1 + b_2)h$$

$$108 = \frac{1}{2}((s) + (2s + 3))(s + 7) \qquad \text{(combine like terms)}$$

$$108 = \frac{1}{2}(3s + 3)(s + 7) \qquad \text{(multiply both sides by 2)}$$

$216 = (3s + 3)(s + 7)$ (expand)
$216 = 3s^2 + 24s + 21$ (subtract 216 from both sides)
$0 = 3s^2 + 24s – 195$ (factor out the G.C.F. of 3)
$0 = 3(s^2 + 8s – 65)$
$3(s^2 + 8s – 65) = 0$

Since the coefficient of the squared term is one, $– 5 \bullet 13 = – 65$, and $– 5 + 13 = 8$, then $3(s^2 + 8s – 65) = 0$ becomes:

$3(s – 5)(s + 13) = 0$ (Zero Product Rule)

$3 = 0$ or $s – 5 = 0$ or $s + 13 = 0$
No Solution $s = 5$ $s = – 13$, not possible

The shorter base is 5 feet.

Ex. 18 If the length of the side of a particular cube is doubled, the volume will increase by 5103 cubic meters. What is the length of each side of the original cube?

Solution:

Let c = the length of the side of the original cube
Then 2s = the length of the side of the new cube.
Recall the formula for the volume of a cube is $V = s^3$.

The volume of the original cube + 5103 = The volume of the new cube

$(c)^3 + 5103 = (2c)^3$ (simplify)
$c^3 + 5103 = 8c^3$ (subtract $8c^3$ from both sides)
$– 7c^3 + 5103 = 0$ (factor out the G.C.F. of – 7)
$– 7(c^3 – 729) = 0$

But $c^3 – 729$ is a difference of cubes with F = c and L = 9. Since $F^3 – L^3 = (F – L)(F^2 + FL + L^2)$, then $– 7(c^3 – 729) = 0$ becomes:

$– 7((c) – (9))((c)^2 + (c)(9) + (9)^2) = 0$ (simplify)
$– 7(c – 9)(c^2 + 9c + 81) = 0$ (Zero Product Rule)

$– 7 = 0$ or $c – 9 = 0$ or $c^2 + 9c + 81 = 0$
No Solution $c = 9$ No real solution

The length of the side of the original cube was 9 meters.