Part of Sect 6.3, Part of Sect 6.4 & Sect 6.5 - Factoring Special Products

Section 6.3 Concept #3 and Section 6.4 Concept #2
Factoring Perfect Square Trinomials

Recall the perfect square trinomials from the previous course:

1) $F^2 + 2FL + L^2 = (F + L)^2$
2) $F^2 - 2FL + L^2 = (F - L)^2$

If a trinomial is a perfect square trinomial, we can use these formulas to factor the trinomial. In order for a trinomial to fit the pattern, the first term has to be a perfect square, the last term has to be a perfect square, the signs in front of the first and last terms has to be positive, and the middle term has to match the appropriate pattern.

Factor the following:

Ex. 1 $9m^2 + 24m + 16$

Solution:
G.C.F. = 1. The first and last terms are perfect squares, so F = 3m and L = 4. The middle term is positive so we want to use pattern #1 to check the middle term: $2FL = 2(3m)(4) = 24m$ which is a match. So,

$$9m^2 + 24m + 16 = (3m + 4)^2$$

Ex. 2 $64y^2 - 144yz + 81z^2$

Solution:
G.C.F. = 1. The first and last terms are perfect squares, so F = 8y and L = 9z. The middle term is negative so we want to use pattern #2 to check the middle term: $-2FL = -2(8y)(9z) = -144yz$ which is a match. So,

$$64y^2 - 144yz + 81z^2 = (8y - 9z)^2$$

Ex. 3 $9y^2 + 30y - 25$

Solution:
G.C.F. = 1. Since there is a subtraction sign in front of the last term, this is not a perfect square trinomials. Thus, we cannot use our patterns on this. In the next section of the notes, we will develop some techniques to factor trinomials that are not perfect squares.

Ex. 4 $-4x^2 + 30x - 25$

<u>Solution:</u>

G.C.F. = – 1. So, $-4x^2 + 30x - 25 = -(4x^2 - 30x + 25)$
The first and last terms are perfect squares, so F = 2x and L = 5.
The middle term is negative so we want to use pattern #2 to check the middle term: $-2FL = -2(2x)(5) = -20x$
which is does not match. So, we cannot factor this any further.
Our answer is $-(4x^2 - 30x + 25)$.

Ex. 5 $50x^4z - 80x^2yz + 32y^2z$

<u>Solution:</u>

G.C.F. = 2z. So, $50x^4z - 80x^2yz + 32y^2z = 2z(25x^4 - 40x^2y + 16y^2)$
The first and last terms are perfect squares, so F = $5x^2$ and L = 4y.
The middle term is negative so we want to use pattern #2 to check the middle term: $-2FL = -2(5x^2)(4y) = -40x^2y$
which is a match. So
$50x^4z - 80x^2yz + 32y^2z = 2z(5x^2 - 4y)^2$

Ex. 6 $\frac{1}{4}r^2 + \frac{1}{3}rs + \frac{1}{9}s^2$

<u>Solution:</u>

G.C.F. = 1. The first and last terms are perfect squares, so F = $\frac{1}{2}r$ and L = $\frac{1}{3}s$. The middle term is positive so we want to use pattern #1 to check the middle term: $2FL = 2\left(\frac{1}{2}r\right)\left(\frac{1}{3}s\right) = \frac{1}{3}rs$ which is a match. So,

$$\frac{1}{4}r^2 + \frac{1}{3}rs + \frac{1}{9}s^2 = \left(\frac{1}{2}r + \frac{1}{3}s\right)^2$$

Section 6.5 Concept #1 Factoring Difference of Squares Binomials.

The third special product from the last chapter was the difference of squares: 3) $F^2 - L^2 = (F - L)(F + L)$
If a binomial is a difference of square, we can use this formula to factor the binomial. In order for a binomial to fit the pattern, the first term has to be a perfect square, the last term has to be a perfect square, and the operation has to be subtraction.

Factor the following:

Ex. 7 $9x^2 - 16y^2$

Solution:

The G.C.F. = 1. The first and last terms are perfect squares, so F = 3x and L = 4y. The operation is subtraction so this does fit pattern #3. Thus,

$$9x^2 - 16y^2 = (3x - 4y)(3x + 4y)$$

Ex. 8 $-\frac{9}{64}x^3 + \frac{16}{49}xy^2$

Solution:

The G.C.F. = – x, so $-\frac{9}{64}x^3 + \frac{16}{49}xy^2 = -x\left(\frac{9}{64}x^2 - \frac{16}{49}y^2\right)$. The first and last terms are perfect squares, so F = $\frac{3}{8}$x and L = $\frac{4}{7}$y. The operation is subtraction so this does fit pattern #3. Thus,

$$-x\left(\frac{9}{64}x^2 - \frac{16}{49}y^2\right) = -x\left(\frac{3}{8}x - \frac{4}{7}y\right)\left(\frac{3}{8}x + \frac{4}{7}y\right)$$

Ex. 9 $36x^2 + 49$

Solution:

The G.C.F. = 1. Although the first and last terms are perfect squares, the operation is addition so this does not fit pattern #3. In fact, the sum of squares is prime in the real numbers.

Ex. 10 $-x^8 + 256$

Solution:

The G.C.F. = – 1. So, $-x^8 + 256 = -(x^8 - 256)$

The first and last terms are perfect squares, so F = x^4 and L = 16. The operation is subtraction so this does fit pattern #3. Thus,

$$-(x^8 - 256) = -(x^4 - 16)(x^4 + 16)$$

But, $x^4 - 16$ is a difference of squares with F = x^2 and L = 4. We can break that down using pattern #3 ($x^4 + 16$ is prime).

$$-\mathbf{(x^4 - 16)}(x^4 + 16) = -\mathbf{(x^2 - 4)(x^2 + 4)}(x^4 + 16)$$

Yet, $x^2 - 4$ is a difference of squares with F = x and L = 2. We can break that down using pattern #3 ($x^2 + 4$ is prime).

$$-\mathit{(x^2 - 4)}(x^2 + 4)(x^4 + 16) = -\mathit{(x - 2)(x + 2)}(x^2 + 4)(x^4 + 16)$$

If we were to look at solving the equation $x^2 - 16 = 0$, we can see that x = 4 and x = – 4 are solutions to the equation since (4) = 16 and $(-4)^2$ = 16. It is for that reason that $x^2 - 16$ can be factored into (x – 4)(x + 4). So, x

$= 4$ is a solution if and only if $x - 4$ is a factor and $x = -4$ is a solution if and only if $x + 4$ is a factor. With the equation $x^2 + 16 = 0$, there is no real number that you can square to get -16 since $(-4)^2 = (4)^2 = 16$. This is why $x^2 + 16$ is prime in the real numbers.

Let's consider the equation $x^3 - 27 = 0$. Since $3^3 = 27$, then $x = 3$ is a solution to the equation. This means that $x - 3$ is a factor of $x^3 - 27$. Similar, With $x^3 + 64 = 0$, $x = -4$ is a solution since $(-4)^3 = -64$. So, $x + 4$ is a factor of $x^3 + 64$. The question then becomes what times $(x - 3)$ is $x^3 - 27$ and what times $(x + 4) = x^3 + 64$. To find out, let's do the division:

<u>Divide the following:</u>

Ex. 11a $(x^3 - 27) \div (x - 3)$

<u>Solution:</u>

$$
\begin{array}{r|l}
 & x^2 + 3x + 9 \\ \hline
x - 3 & x^3 \qquad\qquad - 27 \\
 & \underline{-x^3 + 3x^2 \qquad\qquad} \\
 & \qquad 3x^2 \qquad - 27 \\
 & \quad \underline{-3x^2 + 9x \qquad} \\
 & \qquad\qquad 9x - 27 \\
 & \qquad\qquad \underline{-9x + 27} \\
 & \qquad\qquad\qquad 0
\end{array}
$$

Thus, $(x^3 - 27) \div (x - 3)$
$= x^2 + 3x + 9$

Ex. 11b $(x^3 + 64) \div (x + 4)$

<u>Solution:</u>

$$
\begin{array}{r|l}
 & x^2 - 4x + 16 \\ \hline
x - 4 & x^3 \qquad\qquad + 64 \\
 & \underline{-x^3 - 4x^2 \qquad\qquad} \\
 & \qquad -4x^2 \qquad + 64 \\
 & \qquad \underline{4x^2 + 16x \qquad} \\
 & \qquad\qquad 16x + 64 \\
 & \qquad\qquad \underline{-16x - 64} \\
 & \qquad\qquad\qquad 0
\end{array}
$$

Hence, $(x^3 + 64) \div (x + 4)$
$= x^2 - 4x + 16$

This means that $x^3 - 27 = (x - 3)(x^2 + 3x + 9)$. If we let $F = x$ and $L = 3$, then we get: $F^3 - L^3 = (F - L)(F^2 + FL + L^2)$
Similarly, $x^3 + 64 = (x + 4)(x^2 - 4x + 16)$. If we substitute F for x and L for 4, we get: $F^3 + L^3 = (F + L)(F^2 - FL + L^2)$
We have now derived two more special products:

<u>The Sum and Difference of Cubes*</u>

4) $F^3 + L^3 = (F + L)(F^2 - FL + L^2)$
5) $F^3 - L^3 = (F - L)(F^2 + FL + L^2)$

* - if the degree of $F^2 - FL + L^2$ and $F^2 + FL + L^2$ is 2, then they are prime in the real numbers.

Section 6.5 Concept #2 Factoring a Sum or Difference of Cubes

If a binomial is a sum or difference of cubes, we can use these formula to factor the binomial. In order for a binomial to fit the pattern, the first term has to be a perfect cube, the last term has to be a perfect cube, and then we need to check the operation to see which pattern we will use.

Factor the following:

Ex. 12 $125x^3 + 64y^3$

Solution:

G.C.F. = 1. The first and last terms are perfect cubes so F = 5x and L = 4y (helpful hint, to find the cube root of a number on your calculator, use the $\sqrt[3]{x}$ key). Since the operation is addition, we will use pattern #4.

$$F^3 + L^3 = (F + L)(F^2 - FL + L^2)$$
$$(5x)^3 + (4y)^3 = (5x + 4y)((5x)^2 - (5x)(4y) + (4y)^2)$$
$$= (5x + 4y)(25x^2 - 20xy + 16y^2)$$

The degree of the resulting trinomial is 2, so it is prime. Our answer is $(5x + 4y)(25x^2 - 20xy + 16y^2)$.

Ex. 13 $8x^3 - 27y^3$

Solution:

G.C.F. = 1. The first and last terms are perfect cubes so F = 2x and L = 3y. Since the operation is subtraction, we will use pattern #5.

$$F^3 - L^3 = (F - L)(F^2 + FL + L^2)$$
$$(2x)^3 - (3y)^3 = (2x - 3y)((2x)^2 + (2x)(3y) + (3y)^2)$$
$$= (2x - 3y)(4x^2 + 6xy + 9y^2)$$

The degree of the resulting trinomial is 2, so it is prime. Our answer is $(2x - 3y)(4x^2 + 6xy + 9y^2)$.

Ex. 14 $8x^3 - 18x$

Solution:

G.C.F. = 2x, so $8x^3 - 18x = 2x(4x^2 - 9)$

Notice this is not a sum or difference of cubes, but a difference of squares with F = 2x and L = 3. Thus,

$$2x(4x^2 - 9) = 2x(2x - 3)(2x + 3)$$

With the special products, the difference of squares takes higher priority over the sum and difference of cubes. So, always check to see if you can use the difference of squares first and then move onto the cubes.

Ex. 15 $64x^6 + y^6$

Solution:

G.C.F. = 1. There is no formula for the sum of squares, but we can factor this using the sum of cubes with $F = 4x^2$ and $L = y^2$. Thus,

$$F^3 + L^3 = (F + L)(F^2 - FL + L^2)$$
$$(4x^2)^3 + (y^2)^3 = (4x^2 + y^2)((4x^2)^2 - (4x^2)(y^2) + (y^2)^2)$$
$$= (4x^2 + y^2)(16x^4 - 4x^2y^2 + y^4)$$

It turns out that $16x^4 - 4x^2y^2 + y^4$ is prime in the real numbers, but we will not prove that fact in this course.
So, our answer is $(4x^2 + y^2)(16x^4 - 4x^2y^2 + y^4)$.

Ex. 16 $64x^6 - y^6$

Solution:
G.C.F. = 1, The first and last terms are perfect squares, so $F = 8x^3$ and $L = y^3$. The operation is subtraction so this does fit pattern #3
$64x^6 - y^6 = (8x^3 - y^3)(8x^3 + y^3)$
But $8x^3 - y^3$ is a difference of cubes and $8x^3 + y^3$ is a sum of cubes with $F = 2x$ and $L = y$. So,

$$F^3 - L^3 = (F - L)(F^2 + FL + L^2)$$
$$(2x)^3 - (y)^3 = (2x - y)((2x)^2 + (2x)(y) + (y)^2)$$
$$= (2x - y)(4x^2 + 2xy + y^2)$$

and $F^3 + L^3 = (F + L)(F^2 - FL + L^2)$

$$(2x)^3 + (y)^3 = (2x + y)((2x)^2 - (2x)(y) + (y)^2)$$
$$= (2x + y)(4x^2 - 2xy + y^2)$$

Both of the resulting trinomials are degree 2 so they are prime.
Hence, $(8x^3 - y^3)(8x^3 + y^3)$
$= (2x - y)(4x^2 + 2xy + y^2)\ (2x + y)(4x^2 - 2xy + y^2)$

It is important to memorize these formulas as soon as possible so you can get comfortable using them. Put them on a flash cards or whatever it takes.

Special Product Formulas

1) $F^2 + 2FL + L^2 = (F + L)^2$
2) $F^2 - 2FL + L^2 = (F - L)^2$
3) $F^2 - L^2 = (F - L)(F + L)$
4) $F^3 + L^3 = (F + L)(F^2 - FL + L^2)$
5) $F^3 - L^3 = (F - L)(F^2 + FL + L^2)$

$F^2 + L^2$ (the sum of squares) cannot be factored in the real numbers.
$F^2 - FL + L^2$ and $F^2 + FL + L^2$ are prime in the real numbers if they are degree 2.