Sect 7.1 - Introduction to Rational Expressions

Concept #1 Definition of a Rational Expression.

Recall that a rational number is any number that can be written as the ratio of two integers where the integer in the denominator can not be 0.

Rational Numbers: { $\frac{a}{b}$ | a and b are integers and b ≠ 0 }

When we discuss "algebraic fractions", we can define them in a similar fashion:

Rational Expression ("algebraic fractions")

A **Rational Expression** is the ratio of two polynomials $\frac{P}{Q}$ such that the polynomial Q in the denominator is not 0 (Q ≠ 0).

Some examples of rational expressions include:

$$\frac{2x-5}{3x+9},\ \frac{5}{11},\ \text{and}\ \frac{9x^2-126xy+49y^2}{x^3-27y^3}$$

Concept #2 Evaluating Rational Expressions

Evaluate the following for the given values of x: $\frac{2x-5}{3x+9}$

Ex. 1a x = 5 Ex. 1b x = – 6

Ex. 1c x = 2.5 Ex. 1d x = – 3

Solution:

a) $\frac{2x-5}{3x+9}$ (replace x by (5))

$= \frac{2(5)-5}{3(5)+9} = \frac{10-5}{15+9} = \frac{5}{24}$

b) $\frac{2x-5}{3x+9}$ (replace x by (– 6))

$= \frac{2(-6)-5}{3(-6)+9} = \frac{-12-5}{-18+9} = \frac{-17}{-9} = \frac{17}{9}$

c) $\frac{2x-5}{3x+9}$ (replace x by (2.5))

$= \frac{2(2.5)-5}{3(2.5)+9} = \frac{5-5}{7.5+9} = \frac{0}{16.5} = 0$

d) $\frac{2x-5}{3x+9}$ (replace x by (– 3))

$= \frac{2(-3)-5}{3(-3)+9} = \frac{-6-5}{-9+9} = \frac{-11}{0}$ undefined

Concept #3 Domain of a Rational Expression

In example one, the rational expression $\frac{2x-5}{3x+9}$ was undefined when $x = -3$. But, $x = -3$ is the only value that will give us division by zero. This means that the rational expression $\frac{2x-5}{3x+9}$ is defined for all real numbers except when $x = -3$. The values for which a rational expression is defined is called the **domain** of the rational expression. Thus, the domain of $\frac{2x-5}{3x+9}$ is $\{ x \mid x \text{ is a real number and } x \neq -3\}$.

Informal Definition of the Domain of an Algebraic Expression
The **domain** of an algebraic expression is the values for which the expression is defined (i.e., it is all the values of x that we can plug into the expression and get a number for an answer).

To Find the Domain of a Rational Expression

1. Set the denominator of the expression equal to zero and solve.
2. The domain is all real numbers *except* those values find in part 1.

Find the domain of the following:

Ex. 2a $\frac{19}{3x}$

Ex. 2b $\frac{t-4}{t+7}$

Ex. 2c $\frac{p^2-9}{p^2-7p+10}$

Ex. 2d $\frac{x^2-3x+2}{2x^2+32}$

Solution:

a) Set the denominator equal to zero and solve:
$3x = 0 \Rightarrow x = 0$
Now, exclude $x = 0$ from the real numbers.
Thus, the domain is $\{ x \mid x \text{ is a real number and } x \neq 0\}$.

b) Set the denominator equal to zero and solve:
$t + 7 = 0 \Rightarrow t = -7$
Now, exclude $t = -7$ from the real numbers.
Thus, the domain is $\{ t \mid t \text{ is a real number and } t \neq -7\}$.

c) Set the denominator equal to zero and solve:
$p^2 - 7p + 10 = 0$ (factor)
$(p - 5)(p - 2) = 0$ (set each factor equal to 0 and solve)
$p - 5 = 0$ or $p - 2 = 0$
$p = 5$ or $p = 2$ Now, exclude those values.
Thus, the domain is $\{ p \mid p \text{ is a real number and } p \neq 2, p \neq 5\}$.

d) Set the denominator equal to zero and solve:
$2x^2 + 32 = 0$ (factor out 2)
$2(x^2 + 16) = 0$ ($x^2 + 16$ is prime)
Set each factor equal to zero and solve:
$2 = 0$ or $x^2 + 16 = 0$
No Solution No real solution since $x^2 \neq -16$
Hence, there are no values to exclude.
Thus, the domain is { x | x is a real number}.

Sometimes the symbol $\mathbb{R}$ is used to denote the set of all real numbers. So, we could have said the domain for part d was $\mathbb{R}$.

Concept #4 Simplifying Rational Expressions to Lowest Terms

To reduce fractions to lowest terms, we first rewrote the numerator and denominator as a product and then divided out the common factors:

$$\frac{36}{20} = \frac{4\bullet 9}{4\bullet 5} = \frac{\cancel{4}\bullet 9}{\cancel{4}\bullet 5} = (1)\frac{9}{5} = \frac{9}{5}.$$

We can do the same thing with rational expressions. We first factor the numerator and denominator to make the numerator and denominator into a product. Let's see how an example works:

Given the following expression, a) factor the numerator and denominator, b) find the domain, and c) simplify:

Ex. 3 $\frac{9x^2 - 4}{9x^2 + 9x - 10}$

Solution:

a) Factor the numerator: $9x^2 - 4 = (3x - 2)(3x + 2)$
Factor the denominator: $9x^2 + 9x - 10 = (3x - 2)(3x + 5)$
Thus, $\frac{9x^2 - 4}{9x^2 + 9x - 10} = \frac{(3x-2)(3x+2)}{(3x-2)(3x+5)}$

b) Setting the denominator equal to zero and solving yields:
$(3x - 2)(3x + 5) = 0$
$3x - 2 = 0$ or $3x + 5 = 0$
$3x = 2$ or $3x = -5$
$x = \frac{2}{3}$ or $x = -\frac{5}{3}$ (exclude from $\mathbb{R}$)
The domain is { x| x is a real number and $x \neq -\frac{5}{3}$, $x \neq \frac{2}{3}$}

c) $$\frac{9x^2 - 4}{9x^2 + 9x - 10} = \frac{(3x-2)(3x+2)}{(3x-2)(3x+5)} = \frac{\cancel{(3x-2)}(3x+2)}{\cancel{(3x-2)}(3x+5)} = \frac{3x+2}{3x+5},\ x \neq -\frac{5}{3},\ \&\ \frac{2}{3}$$

It is important to emphasis that $\frac{9x^2-4}{9x^2+9x-10} = \frac{3x+2}{3x+5}$ only if $x \neq -\frac{5}{3}$ and $x \neq \frac{2}{3}$. Though $\frac{3x+2}{3x+5}$ is only undefined at $-\frac{5}{3}$, the original function $\frac{9x^2-4}{9x^2+9x-10}$ was undefined at $x \neq -\frac{5}{3}$ and $x \neq \frac{2}{3}$. We are only allowed to reduce $\frac{3x-2}{3x-2}$ to one provided that $3x - 2 \neq 0$. This leads to our fundamental principle of rational expressions:

Fundamental Principle of Rational Expressions
Let p, q, and r be polynomials such that $q \neq 0$ and $r \neq 0$, then

$$\frac{p \bullet r}{q \bullet r} = \frac{p}{q}(1) = \frac{p}{q}$$

Simplify the following:

Ex. 4 $\frac{76a^5}{24a^3}$

Solution:
Since the numerator and denominator are already a product, we can reduce directly:

$$\frac{76a^5}{24a^3} = \frac{19a^2(4a^3)}{6(4a^3)} = \frac{19a^2}{6}.$$

Ex. 5 $\frac{4x-12}{4x}$

Solution:
We cannot reduce the 4x's since the numerator is not written as a product. To write it as a product, we need to factor:

$$4x - 12 = 4(x - 3)$$

Thus, $\frac{4x-12}{4x} = \frac{4(x-3)}{4(x)} = \frac{(x-3)(4)}{(x)(4)} = \frac{x-3}{x}$.

Ex. 6 $\frac{2T^2+6T+4}{4T^2-12T-16}$

Solution:
We need to factor both the numerator and denominator to write them as a product:
Numerator: $2T^2 + 6T + 4 = 2(T^2 + 3T + 2) = 2(T + 2)(T + 1)$
Denominator: $4T^2 - 12T - 16 = 4(T^2 - 3T - 4) = 4(T - 4)(T + 1)$

Thus,

$$\frac{2T^2+6T+4}{4T^2-12T-16} = \frac{2(T+2)(T+1)}{4(T-4)(T+1)} = \frac{(T+2)[2(T+1)]}{2(T-4)[2(T+1)]} = \frac{T+2}{2(T-4)}.$$

Ex. 7 $\frac{x^4+18x^2+81}{x^2+9}$

Solution:

We need to factor both the numerator and denominator to write them as a product:

Numerator: $F = x^2$, $L = 9$, $2FL = 2(x^2)(9) = 18x^2$

perfect square trinomial

$x^4 + 18x^2 + 81 = (x^2 + 9)^2 = (x^2 + 9)(x^2 + 9)$

But, the sum of squares is prime.

Denominator: sum of squares is prime

Thus, $\frac{x^4+18x^2+81}{x^2+9} = \frac{(x^2+9)(x^2+9)}{1(x^2+9)} = \frac{x^2+9}{1} = x^2 + 9.$

Ex. 8 $\frac{4x^2-9}{16x^4-24x^3+54x-81}$

Solution:

We need to factor both the numerator and denominator to write them as a product:

Numerator: $4x^2 – 9 = (2x – 3)(2x + 3)$

Denominator: $16x^4 – 24x^3 + 54x – 81$

$= (16x^4 – 24x^3) + (54x – 81)$

$= 8x^3(2x – 3) + 27(2x – 3)$

$= (2x – 3)(8x^3 + 27)$

$= (2x – 3)(\mathbf{[2x]^3 + [3]^3})$

$= (2x – 3)\mathbf{(2x + 3)(4x^2 – 6x + 9)}$

Thus,

$$\frac{4x^2-9}{16x^4-24x^3+54x-81} = \frac{1(2x-3)(2x+3)}{(4x^2-6x+9)(2x-3)(2x+3)} = \frac{1}{4x^2-6x+9}.$$

Concept #5 Simplifying a Ratio of – 1

In arithmetic, we saw that number divide by its opposite is – 1. This is not quite so obvious to see in algebra, but if we write our expressions in order of descending powers and factor out a – 1 if the leading term is negative, then the process will work more smoothly.

Simplify:

Ex. 9 $\frac{1-a}{7a^2-7}$

Solution:

We need to factor both the numerator and denominator to write them as a product:

Numerator: $1 - a = -a + 1 = -1(a - 1)$

Denominator: $7a^2 - 7 = 7(a^2 - 1) = 7(a + 1)(a - 1)$

Hence,

$$\frac{1-a}{7a^2-7} = \frac{-1(a-1)}{7(a+1)(a-1)} = \frac{-1}{7(a+1)} = -\frac{1}{7(a+1)}$$

Ex. 10 $\frac{125b^3-27a^3}{9a^2+15ab+25b^2}$

Solution:

We need to factor both the numerator and denominator to write them as a product:

Numerator: Write in order of descending powers with respect to a:

$125b^3 - 27a^3 = -27a^3 + 125b^3 = -1(27a^3 - 125b^3)$

$= -1([3a]^3 - [5b]^3) = -(3a - 5b)(9a^2 + 15ab + 25b^2)$

Denominator: $9a^2 + 15ab + 25b^2$ is prime:

$$\frac{125b^3-27a^3}{9a^2+15ab+25b^2} = \frac{-1(3a-5b)(9a^2+15ab+25b^2)}{1(9a^2+15ab+25b^2)} = \frac{-1(3a-5b)}{1} = -1(3a - 5b)$$

Ex. 11 $\frac{6x^2+23x+20}{2x^2+5x-12}$

Solution:

We need to factor both the numerator and denominator to write them as a product:

Numerator: $6x^2 + 23x + 20 = (3x + 4)(2x + 5)$

Denominator: $2x^2 + 5x - 12 = (2x - 3)(x + 4)$

Thus, $\frac{6x^2+23x+20}{2x^2+5x-12} = \frac{(3x+4)(2x+5)}{(2x-3)(x+4)}$ cannot be reduce. So our answer is: $\frac{(3x+4)(2x+5)}{(2x-3)(x+4)}$.