Sect 8.2 - Introduction to Functions

Concept #1 Definition of a function

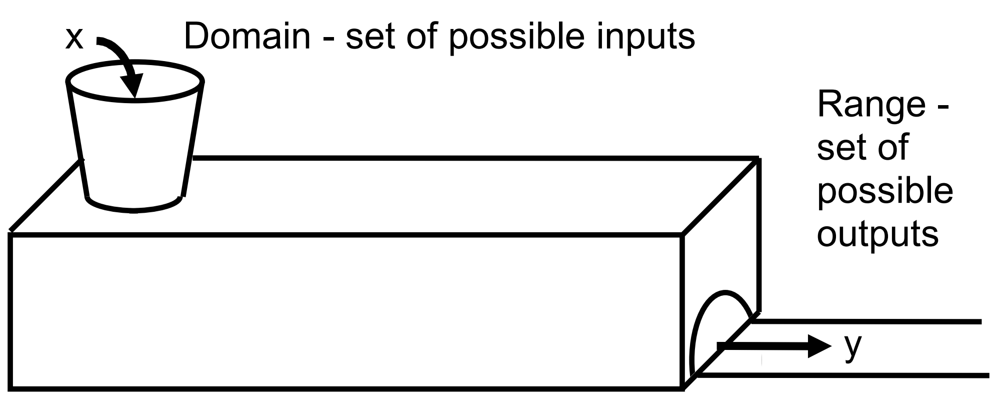
A function is a special type of a relation. With relations, it is possible to have a relation where one value in the domain corresponds to two or more values in the range. Recall the relation in example #2 from the previous section:

{(85˚, 135), (97˚, 236), (92˚,185), (94˚, 201), (97˚, 247)}

In this example, the value of 97˚ in the domain corresponded to both values of 236 and 247 in the range. With functions, this is not allowed to happen. Every element, x, in the domain can only correspond to one element, y, in the range. We cannot have one element in the domain corresponding to more than one element in the range. Thus, the relation in example #2 from the previous section is not a function. Now recall example #3 from the previous section:

{(California, 55), (Texas, 34), (New York, 31), (Florida, 27), (Illinois, 21), (Pennsylvania, 21)}

Since each state corresponds to one number (the number of electoral votes), then the relation is a function. There is no state that has two different numbers of electoral votes. The domain and range of a function is found in the same way we find the domain and range of a relation. Another way to think of a function is think of it as a machine. We put stuff into the machine and it produces an output. The set of all possible inputs is called the domain of the function and the set of possible outputs is called the range of the function. If we put a certain input into our machine, we want to get one output. If we get more than one possible output from a single input, our machine is broken and hence it is not a function.



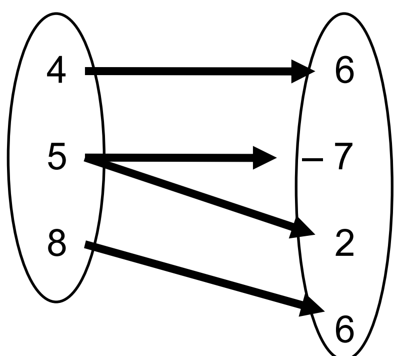
Think of a machine that produces chocolate chip cookies. If we input the ingredients x, we should get chocolate chip cookies as our output y, so our machine would be a function. We can even change x (the ingredients), and we would still get chocolate chip cookies (there is more than one recipe for chocolate chip cookies). Now suppose that there was a machine that when you input a particular set of ingredients x, sometimes you get chocolate chip cookies, but other times you get creamed spinach as your output y. That machine would not be a function since you are getting two different outputs from the same input. Now, let's define a relation and a function more formally.

A **relation** is a correspondence such that each member in the **domain** (x-value) corresponds to at least one member in the **range** (y-value).

A **function** is a correspondence such that each member in the **domain** (x-value) corresponds to exactly one member in the **range** (y-value). We say "y is a function of x."

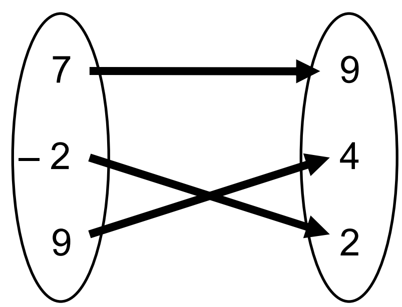
Determine whether the following relation is a function:

Ex. 2



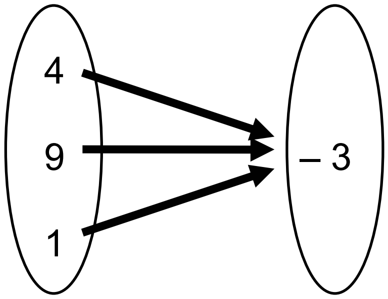
Solution:
This relation is defined by the set:
{(4, 6), (5, – 7), (5, 2), (8, 6)}
When x = 5, there are two possible outputs y, – 7 and 2. Thus, this relation is not a function.

Ex. 3



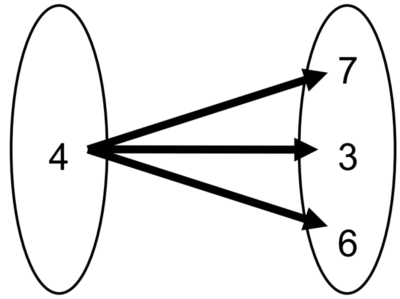
Solution:
This relation is defined by the set:
{(7, 9), (– 2, 2), (9, 4)}
When x = 7, the only possible output is y = 9. When x = – 2, the only possible output is y = 2. When x = 9, the only possible output is y = 4. Since each value x in the domain has only one corresponding y value in the range, the relation is a function.

Ex. 4



<u>Solution:</u>
This relation is defined by the set:
{(4, – 3), (9, – 3), (1, – 3)}
When x = 4, the only possible output is y = – 3. When x = 9, the only possible output is y = – 3. When x = 1, the only possible output is y = – 3. Since each value x in the domain has only one corresponding y value in the range, the relation is a function.

Ex. 5



<u>Solution:</u>
This relation is defined by the set:
{(4, 7), (4, 3), (4, 6)}
When x = 4, there are three possible outputs for y: 7, 3, and 6. Thus, this relation is not a function.

Ex. 6 {(1, 3), (2, 4), (3, 3)}

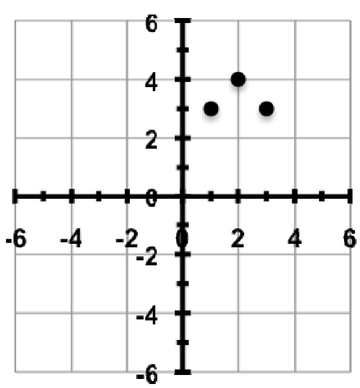
<u>Solution:</u>
Each member in the domain corresponds to exactly to one member in the range, so this relation is a function.

Ex. 7 {(3, 1), (4, 3), (3, 2)}

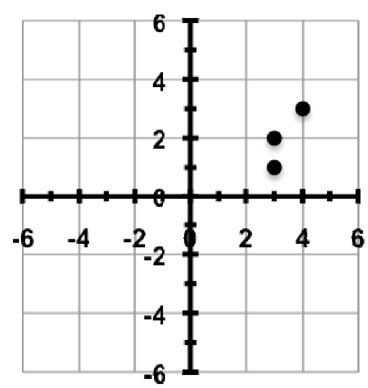
<u>Solution:</u>
The number 3 in the domain corresponds to both 1 and 2 in the range, so the relation is not a function.

We can also view the ordered pairs in the relations in examples #6 and #7 as points on a graph. If we were to plot these points, we would get:

Ex. 6 Function



Ex. 7 Not a function



Notice that every vertical line that intersects the graph of the function in #6 at most one time. Whereas there is a vertical line, namely x = 3, that intersects the graph of the relation in 7 twice. This means that a relation is not a function if there is a vertical line that intersects its graph more than once. This is known as the vertical line test.

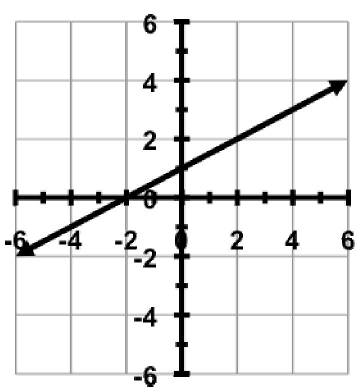
Concept #2 Vertical line Test

<u>Vertical line test for functions</u>

If there exists a vertical line that passes through more than one point on the graph of f(x), then f(x) is not a function.

<u>Given the graph below, a) Is it a function? b) If it is linear, write a down the linear equation. c) State the domain and range:</u>

Ex. 8



<u>Solution:</u>

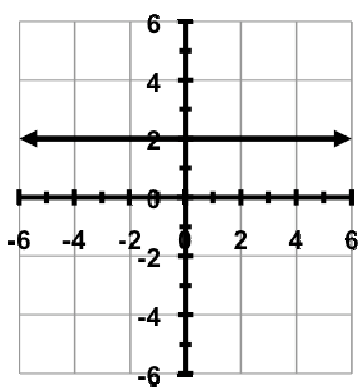
a) Yes, since every vertical line passes through at most one point.

b) From (– 2, 0) to (0, 1), rise 1 unit and run 2 units so,

$m = \frac{\text{"rise"}}{\text{"run"}} = \frac{1}{2}$. The y-intercept is (0, 1), b = 1. Thus, $y = \frac{1}{2}x + 1$

c) Domain: (– ∞, ∞)
Range: (– ∞, ∞)

Ex. 9



<u>Solution:</u>

a) Yes, since every vertical line passes through at most one point.

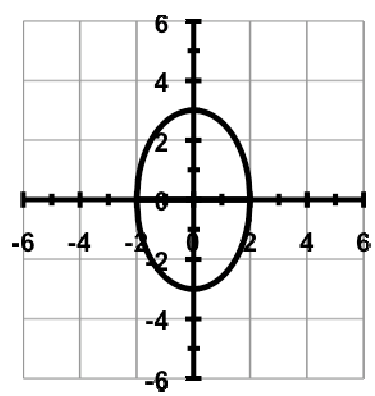
b) The equation of a horizontal line is in the form = y = #,

so the equation is

so y = 2.

c) Domain: (– ∞, ∞)
Range: {2}

Ex. 10



<u>Solution:</u>

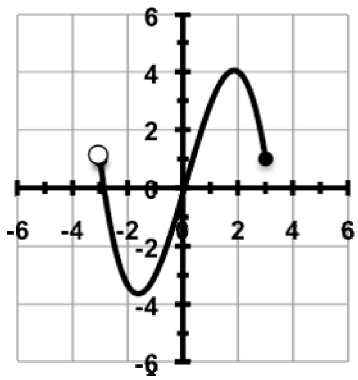
a) No, there is a vertical line (x = 1) that passes through two points.

b) This is not a linear equation. We will write N/A.

c) Domain: [– 2, 2] or
{x | – 2 ≤ x ≤ 2}

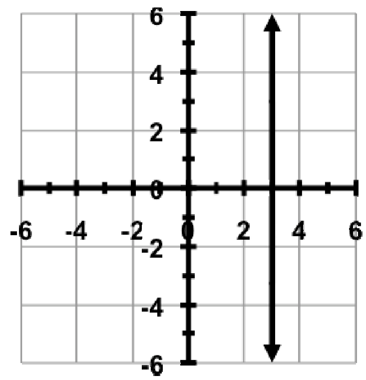
Range: [– 2, 4] or
{y | – 2 ≤ y ≤ 4}

Ex. 11



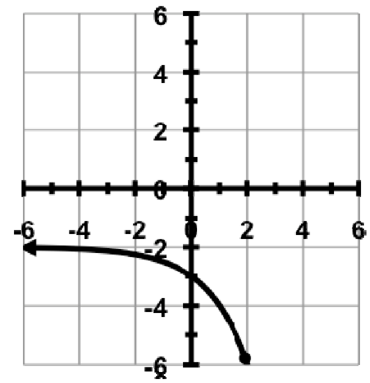
Solution:
a) Yes, since every vertical line passes through at most one point.
b) This is not a linear equation. We will write N/A.
c) Domain: (– 3, 3] or $\{x \mid -3 < x \le 3\}$
Range: [– 3.6, 4] or $\{y \mid -3.6 \le y \le 4\}$

Ex. 12



Solution:
a) No, there is a vertical line (x = 3) that passes through an infinite number of points.
b) The equation of a vertical line is in the form x = #, so the equation is x = 3.
c) Domain {3}
Range: (– ∞, ∞)

Ex. 13



Solution:
a) Yes, since every vertical line passes through at most one point.
b) This is not a linear equation. We will write N/A.
c) Domain: (– ∞, 2] or $\{x \mid x \le 2\}$
Range: [– 6, – 2] or $\{y \mid -6 \le y \le -2\}$

Concept #3 Function Notation.

Evaluating functions works the same as evaluating expressions; we replace the variables by the numbers given and follow the order of operations to simplify:

Evaluate the $x^2 - 6x + 3$ for the given values below:

Ex. 14a x = – 3 Ex. 14b x = 2
Ex. 14c x = 0

Solution:

a) Replace x by – 3 and simplify:
$x^2 - 6x + 3 = (-3)^2 - 6(-3) + 3 = 9 - 6(-3) + 3$
$= 9 + 18 + 3 = 30.$

b) Replace x by 2 and simplify:
$x^2 - 6x + 3 = (2)^2 - 6(2) + 3 = 4 - 6(2) + 3$
$= 4 - 12 + 3 = -5.$

c) Replace x by 0 and simplify:
$x^2 - 6x + 3 = (0)^2 - 6(0) + 3 = 0 - 6(0) + 3$
$= 0 - 0 + 3 = 3.$

In algebra, we can ask the question in example fourteen a different way using what we call function notation. We let f(x) equal to the expression we are evaluating and to evaluate the expression for x = a, we say find f(a). In other words, if $f(x) = x^2 + 3x$ and we are asked to find f(4), that means evaluate $x^2 + 3x$ for x = 4. Let's rewrite example fourteen using this notation:

Given $f(x) = x^2 - 6x + 3$, find:

Ex. 15a f(– 3) Ex. 15b f(2)
Ex. 15c f(0)

Solution:
Replace x in the expression by the given value:

a) $f(-3) = (-3)^2 - 6(-3) + 3 = 9 - 6(-3) + 3$
$= 9 + 18 + 3 = 30.$

b) $f(2) = (2)^2 - 6(2) + 3 = 4 - 6(2) + 3$
$= 4 - 12 + 3 = -5.$

c) $f(0) = (0)^2 - 6(0) + 3 = 0 - 6(0) + 3$
$= 0 - 0 + 3 = 3.$

In our function notation, f(x) does not mean f times x; it means that f is a function of x. Functions can be evaluated not only at numerical values, but also at algebraic expression. Consider the following example.

Given $g(x) = 3x^2 - 5x$, find:

Ex. 16a g(– 4) Ex. 16b g(w)
Ex. 16c g(– r) Ex. 16d g(x – 2b)

Solution:

a) Replace x by – 4:
$g(\mathbf{-4}) = 3(\mathbf{-4})^2 - 5(\mathbf{-4}) = 3(16) - 5(-4) = 48 + 20 = 68$

b) Replace x by w:
$g(\mathbf{w}) = 3(\mathbf{w})^2 - 5(\mathbf{w}) = 3w^2 - 5w$

c) Replace x by – r:
$g(\mathbf{-r}) = 3(\mathbf{-r})^2 - 5(\mathbf{-r}) = 3r^2 + 5r$

d) Replace x by x – 2b:
$g(\mathbf{x - 2b}) = 3(\mathbf{x - 2b})^2 - 5(\mathbf{x - 2b})$
$= 3(x^2 - 4xb + 4b^2) - 5(x - 2b) = 3x^2 - 12xb + 12b^2 - 5x + 10b$

Given r = {(3, 5), (2, – π), (– 4, – 1)}, find:

Ex. 17a r(2) Ex. 17b r(3)
Ex. 17c r(– 4) Ex. 17d r(5)

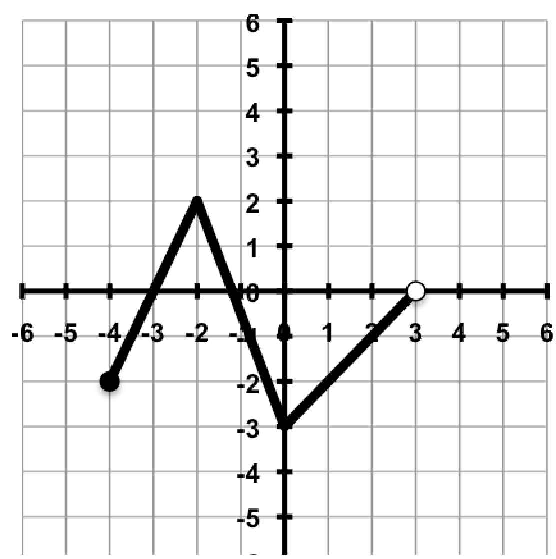
Solution:

a) Since x = 2 corresponds to – π, then r(2) = – π.
b) Since x = 3 corresponds to 5, then r(3) = 5.
c) Since x = – 4 corresponds to – 1, then r(– 4) = – 1.
d) There is no first component of 5 in the function r, so r(5) is undefined.

Concept #4 Finding Function Values from a Graph

We can evaluate a function by looking at the points on the graph. If we are given the graph of f and asked to find f(a), we locate the point on the graph with x-coordinate of a. The y-coordinate of that point is the function value f(a).

Given the graph of f below, find:



Ex. 18a f(– 4)

Ex. 18b f(0)

Ex. 18c f(2)

Ex. 18d Solve f(x) = 2

Ex. 18e Solve f(x) = 0

Ex. 18f State the domain of f.

Ex. 18g State the range of f.

Solution:

a) The point that has x-coordinate of – 4 is (– 4, – 2). Thus, f(– 4) = – 2.

b) The point that has x-coordinate of 0 is (0, – 3). Hence, f(0) = – 3.

c) The point that has x-coordinate of 2 is (2, – 1). So, f(2) = – 1.

d) To solve f(x) = 2, we need to find the point(s) that have y-coordinate of 2. The only point that has that y-coordinate is (– 2, 2). Thus, x = – 2, so the solution is {– 2}.

e) To solve f(x) = 0, we need to find the point(s) that have y-coordinate of 0. There are two points that have that y-coordinate: (– 3, 0) & (≈ – 1.2, 0). We do not count the point (3, 0) since there is an open circle on the graph at that point which means that point is not part of the graph. Thus, x = – 3 and x ≈ – 1.2, so the solutions are {– 3, – 1.2}.

f) The x-values start at x = – 4 and end right before x = 3. So, the domain is [– 4, 3).

g) The y-values start at y = – 3 and end at y = 2. So, the range is [– 3, 2].

Concept #5 The Domain of the Function

If a function is written as an equation in the form y = f(x), then the domain of the function is all real numbers that can be substituted in for x to produce a real number for y. If a value of x is substituted that makes the function undefined, then that value must be excluded from the domain. Some of the ways a function can be undefined is if the substituted value of x produces division by zero or the square root of a negative number.

<u>Find the domain of the following:</u>

19a) $g(x) = 7x + 5$ 19b) $h(x) = \frac{7}{2x-5}$

19c) $f(x) = \sqrt{9-2x}$ 19d) $p(x) = \frac{5x-8}{6x^2+11x-10}$

19e) $h(r) = \frac{r^3+5}{16r^2+25}$ 19f) $v(w) = \frac{7-w}{\sqrt{2w-8}}$

<u>Solution:</u>

a) Since 7x + 5 is defined for any real number, the domain is (– ∞, ∞).

b) The function is undefined when the denominator is 0. Thus,

$2x - 5 \neq 0$

$2x \neq 5$

$x \neq 2.5$ So, the domain is (– ∞, 2.5) U (2.5, ∞).

c) The function is undefined when the expression under the square root sign is negative. Thus, for f to be defined,

$9 - 2x \geq 0$

$-2x \geq -9$ (switch the inequality sign when multiplying or dividing all sides by a negative number)

$x \leq 4.5$ Hence, the domain is $(-\infty, 4.5]$.

d) The function is undefined when the denominator is 0. Thus,

$6x^2 + 11x - 10 \neq 0$ (factor)

$(3x - 2)(2x + 5) \neq 0$ (set each factor ≠ 0 and solve)

$3x - 2 \neq 0$ $\quad$ $2x + 5 \neq 0$

$3x \neq 2$ $\quad$ $2x \neq -5$

$x \neq \frac{2}{3}$ $\quad$ $x \neq -2.5$

So, the domain is $(-\infty, -2.5) \cup (-2.5, \frac{2}{3}) \cup (\frac{2}{3}, \infty)$.

e) The function is undefined when the denominator is 0. Thus,

$16x^2 + 25 \neq 0$

But, the sum of squares is prime in the real numbers so we will not be able to factor. In fact, $16x^2 \geq 0$ for all real numbers, which means $16x^2 + 25 \geq 25$ for all real numbers. The function is then defined for all real numbers. The domain is $(-\infty, \infty)$.

f) The function is undefined when the denominator is 0 or when the expression under the square root is negative, Hence,

$2w - 8 > 0$

$2w > 8$

$w > 4$.

The domain is $(4, \infty)$.