Sect 9.4 - Solving Systems of Linear Equations by Determinants

Concept #1 Introduction to Determinants

Many times in Algebra, we just used the numbers from the problem and performed the necessary computations. We can do the same thing when we are solving systems of equations. We can use what are called a **matrices** to represent the system of equations. For example:

System of Equations	Matrix	
$5x - 3y = 13$ $4x + y = 7$	$\left[\begin{array}{cc\|c} 5 & -3 & 13 \\ 4 & 1 & 7 \end{array}\right]$	Two rows Three columns
$4a + b - c = 6$ $7a + b - 8c = 2$ $4a - 7b + 5c = -1$	$\left[\begin{array}{ccc\|c} 4 & 1 & -1 & 6 \\ 7 & 1 & -8 & 2 \\ 4 & -7 & 5 & -1 \end{array}\right]$	Three rows Four columns

Since the first matrix has two rows and three columns, it is called 2×3 matrix, while the second matrix is 3×4 matrix. We always classify our matrices by first listing the number of rows and then the number of columns. If a matrix has the same number of rows and columns, it is called a **square matrix**. Every square matrix has a real number associated with it called its **determinant**. The determinant is defined as follows for 2×2 matrices:

2×2 Determinants

The determinant for 2×2 matrix $\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$ is denoted as $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ and is defined as: $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1$.

It is almost like cross multiplication except we are subtracting instead of setting the products equal.

Evaluate the following:

Ex. 1 $\begin{vmatrix} 5 & -3 \\ 4 & 1 \end{vmatrix}$

Solution:

$$\begin{vmatrix} 5 & -3 \\ 4 & 1 \end{vmatrix} = 5(1) - 4(-3) = 5 + 12 = 17.$$

Ex. 2 $\begin{vmatrix} 3 & 2 \\ 2 & -3 \end{vmatrix}$

Solution:

$$\begin{vmatrix} 3 & 2 \\ 2 & -3 \end{vmatrix} = 3(-3) - 2(2) = -9 - 4 = -13.$$

Ex. 3 $\begin{vmatrix} 3 & 2 \\ -7 & -5 \end{vmatrix}$

Solution:

$$\begin{vmatrix} 3 & 2 \\ -7 & -5 \end{vmatrix} = 3(-5) - (-7)(2) = -15 + 14 = -1 .$$

Concept #3 Cramer's Rule for Solving a System of Two Linear Equations in Two Variables.

Cramer's Rule for 2x2 Systems

For the system:

$a_1x + b_1y = c_1$ (the equations must be in standard form)

$a_2x + b_2y = c_2$

We can first calculate the following determinants:

$$D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0 \qquad D_x = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix} \qquad D_y = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$$

Then, the solution to the system is

$$x = \frac{D_x}{D} \quad \text{and} \quad y = \frac{D_y}{D}$$

If D = 0, then Cramer's Rule does not apply. The system is either inconsistent or dependent. We must then use another technique to solve the system.

To find D_x, replace the coefficients for x in the determinant by the constant term. To find D_y, replace the coefficients for y in the determinant by the constant term.

To see where Cramer's Rule comes from, let us solve the following system using the addition method.

Solve using addition:

Ex. 5 $a_1x + b_1y = c_1$
$a_2x + b_2y = c_2$

Solution:

Let's first eliminate x. $-a_2$•eqn. #1 + a_1•eqn. #2:

$-a_2 \bullet 1)$ $\quad -a_1a_2x - a_2b_1y = -a_2c_1$
$a_1 \bullet 2)$ $\quad \underline{a_1a_2x + a_1b_2y = a_1c_2}$

$a_1b_2y - a_2b_1y = a_1c_2 - a_2c_1$ (factor out y)

$(a_1b_2 - a_2b_1)y = a_1c_2 - a_2c_1$ (solve for y)

$y = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$

But $a_1c_2 - a_2c_1 = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} = D_y$ and $a_1b_2 - a_2b_1 = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = D$.

So, $y = \frac{D_y}{D}$.

Now, let's eliminate y. b_2•eqn. #1 + $-b_1$•eqn. #2:

$b_2 \bullet 1)$ $\quad a_1b_2x + b_1b_2y = b_2c_1$
$-b_1 \bullet 2)$ $\quad \underline{-a_2b_1x - b_1b_2y = -b_1c_2}$

$a_1b_2x - a_2b_1x = b_2c_1 - b_1c_2$ (factor out x)

$(a_1b_2 - a_2b_1)x = b_2c_1 - b_1c_2$ (solve for x)

$x = \frac{b_2c_1 - b_1c_2}{a_1b_2 - a_2b_1}$

But $b_2c_1 - b_1c_2 = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix} = D_x$ and $a_1b_2 - a_2b_1 = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = D$.

So, $x = \frac{D_x}{D}$.

Solve using Cramer's Rule:

Ex. 6 $5x - 3y = 13$
$4x + y = 7$

Solution:

$D = \begin{vmatrix} 5 & -3 \\ 4 & 1 \end{vmatrix} = 5(1) - 4(-3) = 5 + 12 = 17.$

$D_x = \begin{vmatrix} 13 & -3 \\ 7 & 1 \end{vmatrix} = 13(1) - 7(-3) = 13 + 21 = 34.$

$D_y = \begin{vmatrix} 5 & 13 \\ 4 & 7 \end{vmatrix} = 5(7) - 4(13) = 35 - 52 = -17.$

$x = \frac{D_x}{D} = \frac{34}{17} = 2$ and $y = \frac{D_y}{D} = \frac{-17}{17} = -1$.
So, the solution is (2, – 1).

Ex. 7 $6x - 4y = 12$
$-9x + 6y = -18$

Solution:

$$D = \begin{vmatrix} 6 & -4 \\ -9 & 6 \end{vmatrix} = 6(6) - (-9)(-4) = 36 - 36 = 0.$$

The system is either inconsistent or dependent.
Let's solve using the addition method:

$$6x - 4y = 12 \quad \text{(multiply by 3)} \quad \Rightarrow \quad 18x - 12y = 36$$
$$-9x + 6y = -18 \quad \text{(multiply by 2)} \quad \Rightarrow \quad -18x + 12y = -36$$

Now, add:

$$\begin{array}{r} 18x - 12y = 36 \\ \underline{-18x + 12y = -36} \\ 0 = 0 \end{array} \quad \text{True}$$

The system is dependent.
So, the solution is $\{ (x, y) \mid 6x - 4y = 12\}$.

Ex. 8 $7x - 3y = 5$
$6x + 5y = 2$

Solution:

$$D = \begin{vmatrix} 7 & -3 \\ 6 & 5 \end{vmatrix} = 7(5) - 6(-3) = 35 + 18 = 53.$$

$$D_x = \begin{vmatrix} 5 & -3 \\ 2 & 5 \end{vmatrix} = 5(5) - 2(-3) = 25 + 6 = 31.$$

$$D_y = \begin{vmatrix} 7 & 5 \\ 6 & 2 \end{vmatrix} = 7(2) - 6(5) = 14 - 30 = -16.$$

$x = \frac{D_x}{D} = \frac{31}{53}$ and $y = \frac{D_y}{D} = \frac{-16}{53} = -\frac{16}{53}$.

So, the solution is $\left(\frac{31}{53}, -\frac{16}{53}\right)$.

Ex. 9 $6x - 8y = 6$
$9x - 12y = 2$

<u>Solution:</u>

$$D = \begin{vmatrix} 6 & -8 \\ 9 & -12 \end{vmatrix} = 6(-12) - (9)(-8) = -72 + 72 = 0.$$

The system is either inconsistent or dependent.

Let's solve using the addition method:

$6x - 8y = 6$ (multiply by 3) $\Rightarrow$ $18x - 24y = 18$

$9x - 12y = 2$ (multiply by -2) $\Rightarrow$ $-18x + 24y = -4$

Now, add:

$$\begin{array}{r} 18x - 24y = 18 \\ \underline{-18x + 24y = -4} \\ 0 = 14 \end{array} \quad \text{False}$$

The system is inconsistent.

So, there is no solution.