Sect 1.5 - Multiplication and Division of Real Numbers

Concept #1 Multiplication of Real Numbers

Consider the following two examples.

Solve

Ex. 1 A tank is losing 3 gallons of water a day. How many gallons will it lose in 5 days?

Solution:

To find the answer, we multiply three by five. Thus, 3(5) = 15. So, the tank will lose 15 gallons.

Now, let's solve these problems using real numbers:

Losing 3 gallons is written as – 3, 5 days is written as 5, and a 15 gallon lost can be written as – 15 gallon. So, the problem can be written as – 3(5) = – 15.

This example shows that a negative number times a positive number yields a negative answer.

Ex. 2 On a savings account, a bank accidentally charges a monthly service fee of $3 for four months. How much did the bank have to adjust the balance to correct the error?

Solution:

To find the answer, we need to multiply 3 and 4: 3(4) = 12. The bank had to add $12 to the account to correct the error.

Now, let's solve this problem using real numbers:

A monthly service fee of $3 can be written as – $3, deleting four transactions can be written as – 4, and adding $12 to the account can be written as 12. So, the problem becomes – 3(– 4) = 12.

This says that a negative number times a negative number is a positive number. So, let's write down the rules:

Multiplication of Real Numbers

1. The product of two real numbers with the same signs is positive.
(– #)•(– #) = + Ans. (+ #)•(+ #) = + Ans.

2. The product of two real numbers with the different signs is negative.
(– #)•(+ #) = – Ans. (+ #)•(– #) = – Ans.

3. The product of any real number and zero is zero.

Simplify:

Ex. 3a $-7(-5)$ Ex. 3b $8.7(-0.056)$

Ex. 3c $-18(3)$ Ex. 3d $0(-6\frac{2}{3})$

Ex. 3e $\left(-7\frac{7}{8}\right)\left(-3\frac{5}{9}\right)$ Ex. 3f $\left(-6\frac{3}{4}\right)\left(4\frac{2}{3}\right)$

Solution:

a) $-7(-5) = 35$ ((– #)•(– #) = + Ans.)

b) $8.7(-0.056) = -0.4872$ ((+ #)•(– #) = – Ans.)

c) $-18(3) = -54$ ((– #)•(+ #) = – Ans.)

d) $0(-6\frac{2}{3}) = 0$

e) $\left(-7\frac{7}{8}\right)\left(-3\frac{5}{9}\right)$ (change to improper fractions)

$= \left(-\frac{63}{8}\right)\left(-\frac{32}{9}\right)$ (reduce)

$= \left(-\frac{7}{8}\right)\left(-\frac{32}{1}\right) = \left(-\frac{7}{1}\right)\left(-\frac{4}{1}\right)$ (simplify)

$= (-7)(-4) = 28$ ((– #)•(– #) = + Ans.)

f) $\left(-6\frac{3}{4}\right)\left(4\frac{2}{3}\right)$ (change to improper fractions)

$= \left(-\frac{27}{4}\right)\left(\frac{14}{3}\right)$ (reduce)

$= \left(-\frac{9}{4}\right)\left(\frac{14}{1}\right)$

$= \left(-\frac{9}{2}\right)\left(\frac{7}{1}\right) = -\frac{63}{2} = -31\frac{1}{2}$ ((– #)•(+ #) = – Ans.)

Concept #2 Exponential Expressions

Simplify the following:

Ex. 4 $25 - 4^2$

Solution:

$25 - 4^2$ (#2-Exponents)
$= 25 - 16$ (#4-subtraction)
$= 9$.

Now, let's try it using the techniques developed in this chapter.

$25 - 4^2 =$ (add the opposite)
$= 25 + -4^2$ (#2-Exponents) Keep in mind our results have to consistent with our prior results, so $-4^2 = -16$)
$= 25 + -16$ (#4-Addition)
$= 9$

This says that $-4^2 = -16$. The negative does not get squared, only the number. If we want to write – 4 the quantity squared, we will need to use parenthesis and write $(-4)^2$.
So, $-4^2 = -4\bullet4 = -16$ and $(-4)^2 = (-4)(-4) = 16$.

Simplify the following:

Ex. 5a $(-3)^2$
Ex. 5b -3^2
Ex. 5c -4^3
Ex. 5d $(-4)^3$
Ex. 5e $(-2)^4$
Ex. 5f -2^4

Solution:

a) $(-3)^2 = (-3)(-3) = 9$.
b) $-3^2 = -3\bullet3 = -9$.
c) $-4^3 = -4\bullet4\bullet4 = -16\bullet4 = -64$.
d) $(-4)^3 = (-4)(-4)(-4) = 16(-4) = -64$.
e) $(-2)^4 = (-2)(-2)(-2)(-2) = 4(-2)(-2) = -8(-2) = 16$.
f) $-2^4 = -2\bullet2\bullet2\bullet2 = -4\bullet2\bullet2 = -8\bullet2 = -16$.

Ex. 6a $(-1)^2$
Ex. 6b $(-1)^3$
Ex. 6c $(-1)^4$
Ex. 6d $(-1)^5$
Ex. 6e $(-1)^{7643}$
Ex. 6f $(-1)^{4652}$

Solution:

a) $(-1)^2 = (-1)(-1) = 1$.
b) $(-1)^3 = (-1)(-1)(-1) = -1$.
c) $(-1)^4 = (-1)(-1)(-1)(-1) = 1$.
d) $(-1)^5 = (-1)(-1)(-1)(-1)(-1) = -1$.

Notice a pattern: $(-\#)^{\text{even power}} = +$ Answer & $(-\#)^{\text{odd power}} = -$ Answer.

e) $(-1)^{7643} = -1$ since 7643 is odd.
f) $(-1)^{4652} = 1$ since 4652 is even.

In general, the product of an even number of negative factors is positive and the product of an odd number of negative factors is negative.

Concept #3 Division of Real Numbers

Two numbers are reciprocals if their product is one. Thus, $\frac{2}{3}$ and $\frac{3}{2}$ are reciprocals since $\frac{2}{3}\bullet\frac{3}{2} = 1$.

In general, if $a \neq 0$, then the **reciprocal** of a is $\frac{1}{a}$.

Division of Real Numbers

Let a and b be real numbers such that $b \neq 0$. Then, $a \div b = a \bullet \frac{1}{b}$.

This means that $28 \div 7 = 4$ is equivalent to $28 \bullet \frac{1}{7} = 4$.

Thus, the rules for dividing two real non-zero numbers is the same as in multiplication:

Division of Non-zero Real Numbers

1. The quotient of two non-zero real numbers with the same signs is positive. (– #) ÷ (– #) = + Ans. (+ #) ÷ (+ #) = + Ans.

2. The quotient of two non-zero real numbers with the different signs is negative. (– #) ÷ (+ #) = – Ans. (+ #) ÷ (– #) = – Ans.

Simplify the following:

Ex. 7a $-32 \div 8$

Ex. 7b $-9.5\overline{)-1.254}$

Ex. 7c $\frac{-15}{-35}$

Ex. 7d $\left(5\frac{1}{4}\right) \div \left(-7\frac{7}{8}\right)$

Ex. 7e $\frac{4.2}{-6.8}$

Ex. 7f $\frac{-6\frac{3}{4}}{-7\frac{1}{2}}$

Solution:

a) $-32 \div 8 = -4$ ((– #) ÷ (+ #) = – Ans.)

b) $-1.254 \div (-9.5) = 0.132$ ((– #) ÷ (– #) = + Ans.)

c) $\frac{-15}{-35} = \frac{15}{35} = \frac{3}{7}$ ((– #) ÷ (– #) = + Ans.)

d) $\left(5\frac{1}{4}\right) \div \left(-7\frac{7}{8}\right)$ (change into improper fractions)

$= \left(\frac{21}{4}\right) \div \left(-\frac{63}{8}\right)$ (multiply by the reciprocal of $-\frac{63}{8}$)

$= \left(\frac{21}{4}\right) \bullet \left(-\frac{8}{63}\right)$ (reduce)

$= \left(\frac{1}{4}\right) \bullet \left(-\frac{8}{3}\right)$

$= \left(\frac{1}{1}\right) \bullet \left(-\frac{2}{3}\right) = -\frac{2}{3}$ ((+ #) • (– #) = – Ans.)

e) $\frac{4.2}{-6.8} = -\frac{4.2}{6.8}$ ((#) ÷ (– #) = – Ans.)

$4.2 \div 6.8 = 0.61764705...$ on a calculator which is messy.

We can write this in fractional form by moving the decimal point one place to the right. (We are multiplying top and bottom by 10)

Thus, $-\frac{4.2}{6.8} = -\frac{42}{68}$ (reduce)

$= -\frac{21}{34}$

f) $\dfrac{-6\frac{3}{4}}{-7\frac{1}{2}} = \left(-6\frac{3}{4}\right) \div \left(-7\frac{1}{2}\right)$ (change into improper fractions)

$= \left(-\frac{27}{4}\right) \div \left(-\frac{15}{2}\right)$ (multiply by the reciprocal of $-\frac{15}{2}$)

$= \left(-\frac{27}{4}\right)\left(-\frac{2}{15}\right)$ (reduce)

$= \left(-\frac{9}{4}\right)\left(-\frac{2}{5}\right)$

$= \left(-\frac{9}{2}\right)\left(-\frac{1}{5}\right) = \frac{9}{10}.$ ((– #) • (– #) = + Ans.)

The following are equivalent:

$-\frac{3}{4} = \frac{-3}{4} = \frac{3}{-4} = -\frac{-3}{-4}$ ($-\frac{3}{4}$ is the most simplified form)

But $\frac{-3}{-4} \neq -\frac{3}{4}$, since $\frac{-3}{-4} = \frac{3}{4}$.

Also, the following are equivalent:

$\frac{3}{4} = -\frac{-3}{4} = -\frac{3}{-4} = \frac{-3}{-4}$ ($\frac{3}{4}$ is the most simplified form)

Some notes on division involving zero:
If a is any real number except for zero, then

A) $\frac{0}{a} = 0$

B) $\frac{a}{0}$ is undefined.

C) $\frac{0}{0}$ is *undetermined.* (It can be equal to any number).

To see why, consider the following:
Any division question can be rephrased as a multiplication question. For example, if we have the problem 24 ÷ ***3*** = ?, we can rephrase the problem as “what do you have to multiply 3 by to get 24:” ***3* • ?** = 24. The answer is eight. In the problem 0 ÷ ***7*** = ?, we can ask “what do you have to multiply 7 by to get 0:” ***7* • ?** = 0. The answer is zero. But in the problem 6 ÷ ***0*** = ?, we run into trouble when we ask “what do you have to multiply 0 by to get 6:” ***0* • ?** = 6. There is no number that works since 0 times any number is 0. The problem is **undefined**.

Concept #4 Order of Operations

Ex. 8 $6(0.7 - 0.2(8))^2 \div (-1.3 - 0.5)$

Solution:

$6(0.7 - 0.2(8))^2 \div (-1.3 - 0.5)$ (#1-parentheses, #3-multiplication)

$= 6(0.7 - 1.6)^2 \div (-1.3 - 0.5)$ (rewrite as addition)

$= 6(0.7 + (-1.6))^2 \div (-1.3 + (-0.5))$ (#1-parentheses, #4-addition)

$= 6(-0.9)^2 \div (-1.8)$ (#2-exponents)

$= 6(0.81) \div (-1.8)$ (#3-multiplication)

$= 4.86 \div (-1.8)$ (#3-division)

$= -2.7$

Ex. 9 $\dfrac{4\bullet(-4)^2 - 4\left(\frac{125}{5} - 8\right)}{-3+3(4\bullet7\bullet1)+(-9\bullet7)}$

Solution:

Let's first work out the numerator:

$4\bullet(-4)^2 - 4\left(\frac{125}{5} - 8\right)$ (#1-parentheses, #3-division)

$= 4\bullet(-4)^2 - 4(25 - 8)$ (#1-parentheses, #4-subtraction)

$= 4\bullet(-4)^2 - 4(17)$ (#2-exponents)

$= 4\bullet(16) - 4(17)$ (#3-multiplication)

$= 64 - 4(17)$ (#3-multiplication)

$= 64 - 68$ (rewrite as addition)

$= 64 + (-68) = -4$ (#4-addition)

Now, let's work the denominator:

$-3 + 3(4\bullet7\bullet1) + (-9\bullet7)$ (#1-parentheses, #3-multiplication)

$= -3 + 3(28\bullet1) + (-9\bullet7)$ (#1-parentheses, #3-multiplication)

$= -3 + 3(28) + (-9\bullet7)$ (#1-parentheses, #3-multiplication)

$= -3 + 3(28) + (-63)$ (#3-multiplication)

$= -3 + 84 + (-63)$ (#4-addition)

$= 81 + (-63) = 18$ (#4-addition)

Thus, $\dfrac{4\bullet(-4)^2 - 4\left(\frac{125}{5} - 8\right)}{-3+3(4\bullet7\bullet1)+(-9\bullet7)} = \dfrac{-4}{18} = -\dfrac{2}{9}$

Ex. 10 $-3.3\sqrt{5-(0.4)^2} \div \left(-\frac{11}{10}\right)\left(\frac{3}{10}\right) - |30-45| \bullet 6 \div 9$

Solution:

Since $\frac{11}{10} = 1.1$ and $\frac{3}{10} = 0.3$, replace the fractions by their decimal equivalents:

$-3.3\sqrt{5-(0.4)^2} \div \left(-\frac{11}{10}\right)\left(\frac{3}{10}\right) - |30-45| \bullet 6 \div 9$

$= -3.3\sqrt{5-(0.4)^2} \div (-1.1)(0.3) - |30-45| \bullet 6 \div 9$

(#1-radical, #2-exponents)

$= -3.3\underline{\sqrt{5-0.16}} \div (-1.1)(0.3) - \underline{|30-45|} \bullet 6 \div 9$

(rewrite as addition inside of the grouping symbols)

$= -3.3\underline{\sqrt{5+(-0.16)}} \div (-1.1)(0.3) - \underline{|30+(-45)|} \bullet 6 \div 9$

(#1-radical & absolute value, #4-addition)

$= -3.3\sqrt{4.84} \div (-1.1)(0.3) - |-15| \bullet 6 \div 9$ (#1-absolute value)

$= -3.3\sqrt{4.84} \div (-1.1)(0.3) - 15 \bullet 6 \div 9$ (#2-exponents)

$= -3.3(2.2) \div (-1.1)(0.3) - 15 \bullet 6 \div 9$ (#3-multiplication)

$= -7.26 \div (-1.1)(0.3) - 15 \bullet 6 \div 9$ (#3-division)

$= 6.6(0.3) - 15 \bullet 6 \div 9$ (#3-multiplication)

$= 1.98 - 15 \bullet 6 \div 9$ (#3-multiplication)

$= 1.98 - 90 \div 9$ (#3-division)

$= 1.98 - 10$ (change to addition, change the sign to the right)

$= 1.98 + (-10)$ (#4-addition)

$= -8.02$

Given r = – 8, evaluate the following:

Ex. 11a r^2 Ex. 11b $-r^2$

Ex. 11c $r^2 - r$ Ex. 11d $-r - |r|$

Solution:

a) $r^2 = (-8)^2 = (-8)(-8) = 64$

b) $-r^2 = -(-8)^2 = -(-8)(-8) = -64$

c) $r^2 - r = (-8)^2 - (-8) = 64 - (-8) = 64 + 8 = 72$

d) $-r - |r| = -(-8) - |-8| = -(-8) - 8 = 8 - 8 = 0$