Sect 2.7 - Solving and graphing inequalities

Concepts #1 & 2 Graphing Linear Inequalities

Definition of a Linear Inequality in One Variable
Let a and b be real numbers such that $a \neq 0$. A **Linear Inequality in One Variable** is an inequality that can written in one of the following forms:

$ax + b > 0$ $ax + b \geq 0$ $ax + b < 0$ $ax + b \leq 0$

List some solutions to the following:

Ex. 1 $x > 3$

Solution:

$4, 7, 6, 8.5, \frac{52}{3}, \sqrt{10}, \ldots$

In fact, any number larger than three is a solution to the inequality.

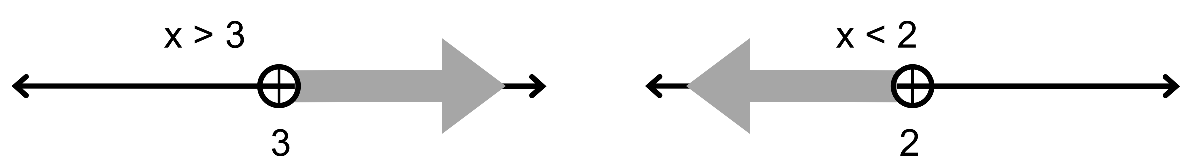
Ex. 2 $x < 2$

Solution:

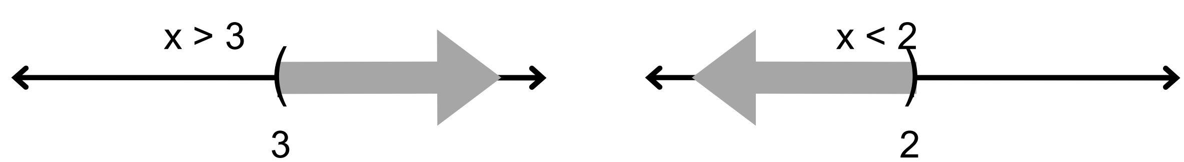
$0, -1, -9.1564, \frac{5}{13}, \sqrt{0.65}, \ldots$

In fact, any number smaller than two is a solution to the inequality.

In both of these examples, there are an infinite number of solutions so it is impossible the write down all the possible numbers. But, we can represent all the solutions to each inequality as a graph on the number line. To represent all the numbers greater than three, we draw an open circle at x = 3 and shaded everything to the right of 3. Likewise, for all the numbers less than two, we draw an open circle at x = 2 and shaded everything to the left of x = 2.



Another version of graphing inequalities uses a parenthesis instead of an open circle. So, in that notation, our graphs would look like:

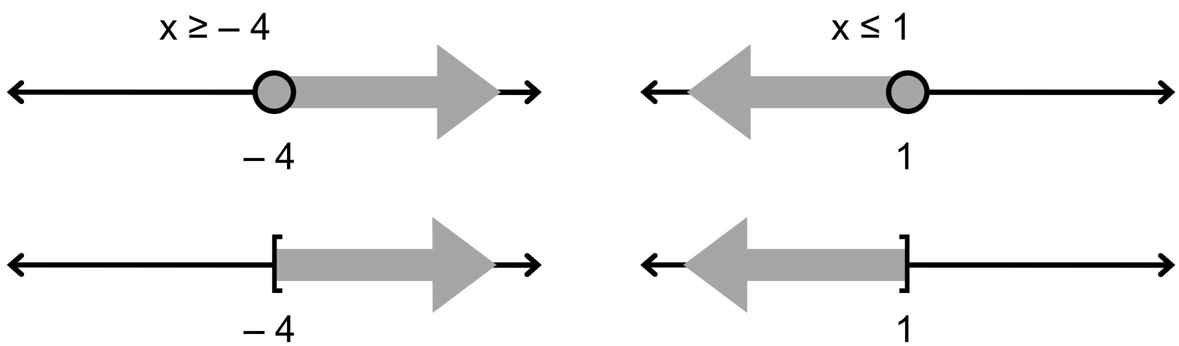


The symbol ≥ represents greater than or equal to and the symbol ≤ represents less than or equal to. This means that the number is included in the solution. To represent this using a graph, we used a filled in circle or a square bracket instead of an open circle or a parenthesis.

Graph the solution to the following:

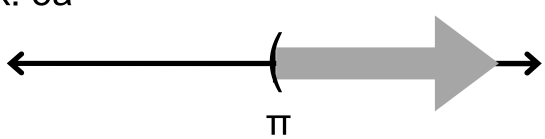
Ex. 3 $x \geq -4$ Ex. 4 $x \leq 1$

To represent all the numbers greater than or equal to negative four, we draw a filled circle or a square bracket at x = – 4 and shaded everything to the right of – 4. Likewise, for all the numbers less than or equal to one, we draw a filled circle or a square bracket at x = 1 and shaded everything to the left of x = 1. A highlighter works well for the shading.

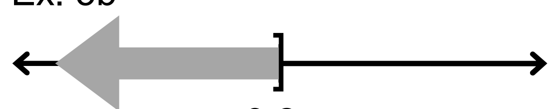


Identify the inequality that is represented by each graph:

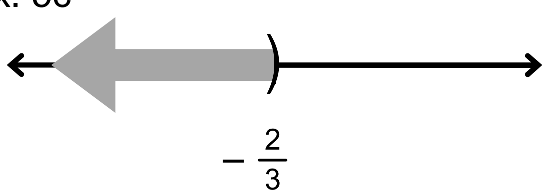
Ex. 5a



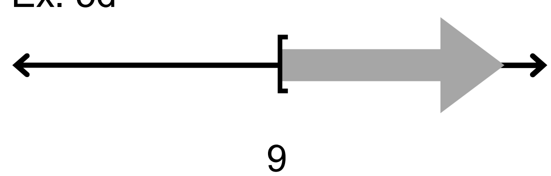
Ex. 5b



Ex. 5c



Ex. 5d



Solution:

a) The solution is all numbers greater than π or $x > \pi$.

b) The solution is all numbers less than or equal to – 3.2 or $x \leq -3.2$.

c) The solution is all numbers less than $-\frac{2}{3}$ or $x < -\frac{2}{3}$.

d) The solution is all numbers greater than or equal to 9 or $x \geq 9$.

A statement that involves more than one inequality is called a compound inequality. In this section, we will examine the type of inequality that indicates that our variable is between two numbers.

<u>Graph the solution to the following:</u>

Ex. 6 $-3 < x \le 7$

<u>Solution:</u>

This means that $x > -3$ and $x \le 7$. In other words, x is between -3 and 7 including 7. We shade the number line from -3 to 7, putting a parenthesis at -3 and a bracket at 7.

$-3 < x \le 7$

Ex. 7 $-\frac{2}{3} \le x < \frac{16}{3}$

<u>Solution:</u>

This means that $x \ge -\frac{2}{3}$ and $x < \frac{16}{3}$. In other words, x is between $-\frac{2}{3}$ and $\frac{16}{3}$ including $-\frac{2}{3}$. We shade the number line from $-\frac{2}{3}$ to $\frac{16}{3}$, putting a bracket at $-\frac{2}{3}$ and a parenthesis at $\frac{16}{3}$.

$-\frac{2}{3} \le x < \frac{16}{3}$

Concept #2 Set-Builder Notation and Interval Notation

Many times we will write our answers in **set-builder notation**. To write a solution to an inequality in set-builder notation, we write

{ x | "the inequality" }

This means: "The set of x such that the inequality"

This notation is very convenient for expressing our answers.

For $x > 3$, our answer is:

$\{ x \mid x > 3\}$

This literally means:
"The set of all x such that x is greater than 3."

For $y \le -2$, our answer is:

$\{ y \mid y \le -2\}$

This literally means
"The set of all y such that y is less than or equal to – 2."

For $-4 \le x < 2$, our answer is:

$\{ x \mid -4 \le x < 2\}$

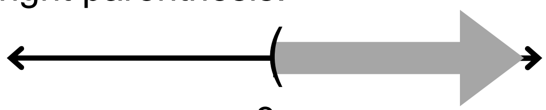
"The set of all x such that x is between – 4 and 2, including – 4."

For $a \ne -7$, our answer is

$\{ a \mid a \ne -7\}$

"The set of all a such that a is not equal to – 7.

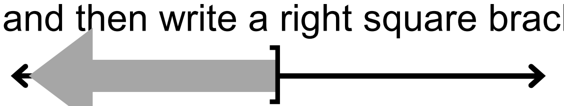
It is also convenient to write our answers in **interval notation**. First, think of the graph of $x > 3$; it extends forever to the right of 3 to infinity. In interval notation, we start by putting a left parenthesis around the endpoint of 3, then we write a comma. Next, we write the infinity symbol ∞ and then write a right parenthesis.



3 $\to \infty$

$(3, \infty)$

The graph of $x \leq 2$ extends forever to the left of 2 to negative infinity and includes 2. In interval notation, we start by putting a left parenthesis around negative infinity $-\infty$, then we write a comma. Next, we write the endpoint of 2 and then write a right square bracket.



$-\infty \leftarrow$ 2

$(-\infty, 2]$

The quantities used with interval notation are always written from smallest to largest (left to right on the number line). So, it is the smaller quantity comma the larger quantity. Parentheses are always used with $-\infty$ (left parenthesis) and ∞ (right parenthesis). A parenthesis is used around the endpoint if it is not included in the solution and a square bracket if it is included in the solution.

<u>Write the solution in set-builder notation and interval notation:</u>

Ex. 8a) $x \leq -4.5$ Ex. 8b) $y > 3.6$

Ex. 8c) $-3 < a < 2$ Ex. 8d) $\frac{3}{8} \leq w \leq \frac{19}{4}$

<u>Solution:</u>

a) Set-Builder Notation: $\{ x \mid x \leq -4.5 \}$.
Interval Notation: $(-\infty, -4.5]$

b) Set-Builder Notation: $\{ y \mid y > 3.6 \}$.
Interval Notation: $(3.6, \infty)$

c) Set-Builder Notation: $\{ a \mid -3 < a < 2 \}$.
Interval Notation: $(-3, 2)$

d) Set-Builder Notation: $\left\{ w \mid \frac{3}{8} \leq w \leq \frac{19}{4} \right\}$.
Interval Notation: $\left[\frac{3}{8}, \frac{19}{4} \right]$.

Concept #3 Addition and Subtraction Properties of Inequality

We can solve inequalities in a similar fashion to how we solved equations. If we add or subtract the same quantity from both sides of an inequality, the inequality symbol will remain the same. These are known as the addition and subtraction properties of inequality:

Addition and Subtraction Properties of Inequality

If a, b, and c are real numbers and if $a < b$, then
$a + c < b + c$ and $a - c < b - c$ are equivalent inequalities to $a < b$.
This also applies for $\leq$, $>$, and $\geq$.

Solve and write your answer in interval notation:

Ex. 9 $x - 9.6 \leq -4.5$

Solution:

$$\begin{array}{l} x - 9.6 \leq -4.5 \\ \underline{\quad +9.6 = +9.6} \\ \quad x \leq 5.1 \end{array}$$

(add 9.6 to both sides)

The solution is $(-\infty, 5.1]$.

Ex. 10 $4x > 3x - \frac{2}{3}$

Solution:

$$\begin{array}{l} 4x > 3x - \frac{2}{3} \\ \underline{-3x = -3x} \\ \quad x > -\frac{2}{3} \end{array}$$

(subtract 3x from both sides)

The solution is $(-\frac{2}{3}, \infty)$.

Concept #4 Multiplication and Division Properties of Inequality

Now, let us examine what happens to the inequality when we multiply both sides by a number. To see this more clearly, we will work with several numerical examples:

Multiply both sides of the inequality by a positive five. Is the resulting inequality true?

Ex. 11 $-4 < 6$

Solution:

$\mathbf{5(}-4\mathbf{)} < \mathbf{5(}6\mathbf{)}$

$-20 < 30$

True.

Ex. 12 $-6 > -8$

Solution:

$\mathbf{5(}-6\mathbf{)} > \mathbf{5(}-8\mathbf{)}$

$-30 > -40$

True.

Hence, if we multiply both sides of an inequality by a positive number, the inequality sign stays the same. The same will hold true for division.

Multiply both sides of the inequality by a negative five. Is the resulting inequality true?

Ex. 13 $-4 < 6$

Solution:

$\mathbf{-5(}-4\mathbf{)} < \mathbf{-5(}6\mathbf{)}$

$20 < -30$

False since $20 > -30$.

Ex. 14 $-6 > -8$

Solution:

$\mathbf{-5(}-6\mathbf{)} > \mathbf{-5(}-8\mathbf{)}$

$30 > 40$

False since $30 < 40$.

Hence, if we multiply both sides of an inequality by a negative number, the inequality sign is reversed. The same will hold true for division.

Multiplication and Division Properties of Inequality

If a, b, and c are real numbers, if $a < b$,

Case 1: and if $c > 0$, then $ac < bc$ and $\frac{a}{c} < \frac{b}{c}$ are equivalent inequalities to $a < b$.

Case 2: and if $c < 0$, then $ac > bc$ and $\frac{a}{c} > \frac{b}{c}$ are equivalent inequalities to $a < b$.

This also applies for ≤, >, and ≥.

This means that whenever we multiply or divide both sides of an inequality by a negative number the inequality sign must be reversed.

Solve and write your answer in interval notation:

Ex. 15 $8(x - 5) \geq 13$

Solution:

$8(x - 5) \geq 13$ (distribute the 8)

$8x - 40 \geq 13$ (add 40 to both sides)

$\underline{\quad + 40 = + 40 \quad}$

$8x \geq 53$

$\frac{8x}{8} \geq \frac{53}{8}$ (divide both sides by 8)

$x \geq 6.625$

So, our answer is $[6.625, \infty)$.

Ex. 16 $\quad -0.17x + 4.2 < 0.3$

<u>Solution:</u>

$$\begin{array}{ll} -0.17x + 4.2 < 0.3 & \text{(subtract 4.2 from both sides)} \\ \underline{\ -4.2 = -4.2} & \\ -0.17x \quad < -3.9 & \end{array}$$

$$\frac{-0.17x}{-0.17} < \frac{-3.9}{-0.17} \quad \text{(divide both sides by } -0.17\text{, and reverse the inequality sign)}$$

$$x > \frac{3.9}{0.17} = \frac{390}{17}$$

Thus, our answer is $\left(\frac{390}{17}, \infty\right)$.

Ex. 17 $\quad -\frac{7}{8}(3x-4) > 6(2x-3) - 3(4x - \frac{11}{3})$

<u>Solution:</u>

$-\frac{7}{8}(3x-4) > 6(2x-3) - 3(4x - \frac{11}{3})$ (distribute)

$-\frac{21}{8}x + \frac{7}{2} > 12x - 18 - 12x + 11$ (combine like terms)

$-\frac{21}{8}x + \frac{7}{2} > -7$ (multiply both sides by 8)

$\frac{8}{1}\left(-\frac{21}{8}x + \frac{7}{2}\right) > 8(-7)$ (distribute)

$\frac{8}{1}\left(-\frac{21}{8}x\right) + \frac{8}{1}\left(\frac{7}{2}\right) > 8(-7)$ (multiply)

$$\begin{array}{ll} -21x + 28 > -56 & \text{(subtract 28 from both sides)} \\ \underline{\ -28 = -28} & \\ -21x \quad > -84 & \end{array}$$

$$\frac{-21x}{-21} > \frac{-84}{-21} \quad \text{(divide both sides by } -21\text{, and reverse the inequality sign)}$$

$$x < 4$$

Hence, our answer is $(-\infty, 4)$.

Concept #5 Solving Inequalities is the form a < x < b

To solve a compound inequality of the form $a < x < b$, we always try to isolate the variable in the middle of the inequality. When we use the addition, subtraction, multiplication, and division properties of inequality, we must apply them to all three sides of the inequality.

<u>Solve and graph the following inequalities. Write the answer in interval notation:</u>

Ex. 18 $\quad -7 < 4x - 5 \le 3$

<u>Solution:</u>

$-7 < 4x - 5 \le 3$ (add 5 to all three sides)

$\underline{+5 = \quad +5 = +5}$

$-2 < 4x \le 8$

$\frac{-2}{4} < \frac{4x}{4} \le \frac{8}{4}$ (divide all three sides by 4)

$-0.5 < x \le 2$

Graph:

Interval Notation: (– 0. 5, 2]

Ex. 19 $\quad -2 < 6 - \frac{2}{3}x \le 4$

<u>Solution:</u>

$-2 < 6 - \frac{2}{3}x \le 4$ (multiply all sides by 3 to clear fractions)

$3(-2) < 3\left(6 - \frac{2}{3}x\right) \le 3(4)$ (distribute)

$3(-2) < 3(6) - 3\left(\frac{2}{3}x\right) \le 3(4)$ (multiply)

$-6 < 18 - 2x \le 12$ (subtract 18 from all three sides)

$\underline{-18 = -18 \quad = -18}$

$-24 < -2x \le -6$

$\frac{-24}{-2} \quad \frac{-2x}{-2} \quad \frac{-6}{-2}$ (divide all three sides by – 2 and reverse the inequalities signs)

$12 > x \ge 3$ (rewrite from smallest to largest)

$3 \le x < 12$

Graph:

Interval Notation: [3, 12)

Concept #6 Applications of Linear Inequalities

Here are some key phrase in translating inequalities:

Inequalities

r is greater than 7	$r > 7$
v exceeds w	$v > w$
4 is greater than or equal to d	$4 \geq d$
g is at least 21	$g \geq 21$
e is less than 6	$e < 6$
b is less than or equal to 9	$b \leq 9$
a is at most 12	$a \leq 12$
9 cannot exceed u	$9 \leq u$
d is no more than x	$d \leq x$
e is no less than 4	$e \geq 4$
r is between 8 and 11	$8 < r < 11$
y is between – 5 and 7 inclusively	$-5 \leq y \leq 7$

<u>**Write an inequality and solve the following:**</u>

Ex. 20 The base of a triangle is fixed at $\frac{5}{6}$ ft. For what heights will the area of the triangle be at most $\frac{59}{51}$ ft^2?

<u>Solution:</u>

The words "at most" mean is less than or equal to. The formula for the area of a triangle is $A = \frac{1}{2}bh$. Since the area is at most $\frac{59}{51}$ ft^2, then $\frac{1}{2}bh \leq \frac{59}{51}$, but b is fixed at $\frac{5}{6}$ ft so,

$$\frac{1}{2}\left(\frac{5}{6}\right)h \leq \frac{59}{51} \qquad \text{(multiply)}$$

$$\frac{\frac{5}{12}h}{\frac{5}{12}} \leq \frac{\frac{59}{51}}{\frac{5}{12}} \qquad \text{(divide both sides by } \tfrac{5}{12}\text{)}$$

$$h \leq \frac{59}{51} \div \frac{5}{12} = \frac{59}{51} \bullet \frac{12}{5} = \frac{59}{17} \bullet \frac{4}{5} = \frac{236}{85} \text{ ft.}$$

So, the height will be at most $\frac{236}{85}$ ft.

Ex. 21 Juanita had scores of 73%, 86%, 82%, and 71% on the first four tests. What must she score on test #5 to have at least a "B" average?

Solution:

The words "at least" mean greater than or equal to. So, to have at least a "B", the average has to be greater than or equal to 80. To find the average of five tests, add the scores and then divide by 5:

Let F = the score on test #5

$$\frac{73+86+82+71+F}{5} \geq 80 \qquad \text{(clear fractions)}$$

$$\frac{5}{1}\left(\frac{73+86+82+71+F}{5}\right) \geq 5 \bullet 80 \qquad \text{(simplify)}$$

$$(73 + 86 + 82 + 71 + F) \geq 400 \qquad \text{(combine like terms)}$$

$$312 + F \geq 400 \qquad \text{(subtract 312 from both sides)}$$

$$\underline{-312 \quad = -312}$$

$$F \geq 88.$$

Thus, she must score at least an 88% on test #5.