Sect 5.3 - Definitions of a^0 and a^{-n}

Concept #1 Definition of a^0.

Let's examine the quotient rule when the powers are equal.

Simplify:

Ex. 1 $\frac{2^5}{2^5}$

Solution:

There are two ways to view this problem. First, any non-zero number divided by itself is 1, so, $\frac{2^5}{2^5} = 1$. But, using the quotient rule, $\frac{2^5}{2^5} = 2^{5-5} = 2^0$. This says that $2^0 = 1$. We can do this same trick with any base except for zero.

Zero Exponents

If a is any non-zero real number, then $a^0 = 1$.

Simplify:

Ex. 2a	3^0	Ex. 2b	$(-3)^0$
Ex. 2c	-3^0	Ex. 2d	$-3x^0$
Ex. 2e	$-3x^2y^0$	Ex. 2f	$-3(x^2y)^0$
Ex. 2g	$-(3x^2y)^0$	Ex. 2h	$(-3x^2y)^0$

Solution:

a) $3^0 = 1$
b) $(-3)^0 = 1$
c) $-3^0 = -(1) = -1$ (the 0 exponent only applies to 3)
d) $-3x^0 = -3(1) = -3$ (the 0 exponent only applies to x)
e) $-3x^2y^0 = -3x^2(1) = -3x^2$. (the 0 exponent only applies to y)
f) $-3(x^2y)^0 = -3(1) = -3$. (the 0 exponent only applies to x^2y)
g) $-(3x^2y)^0 = -(1) = -1$. (the 0 exponent only applies to $3x^2y$)
h) $(-3x^2y)^0 = (1) = 1$.

Concept #2 Definition of a^{-n}

Let's examine the quotient rule when the power in the denominator is larger than the power in the numerator.

Simplify:

Ex. 3 $\frac{2^5}{2^8}$

Solution:

There are two ways to view this problem. First, $2^5 = 32$ and $2^8 = 256$, then $\frac{2^5}{2^8} = \frac{32}{256}$ which reduces to $\frac{1}{8}$. But, $\frac{1}{8} = \frac{1}{2^3}$,so, $\frac{2^5}{2^8} = \frac{1}{2^3}$.

But, using the quotient rule, $\frac{2^5}{2^8} = 2^{5-8} = 2^{-3}$. This says that $2^{-3} = \frac{1}{2^3}$. We can do this same trick with any base except for zero.

Also, $\frac{1}{4^{-2}} = 4^2 = 16$ since $\frac{1}{4^{-2}} = 1 \div 4^{-2} = 1 \div \frac{1}{4^2} = 1 \bullet 4^2 = 4^2 = 16$

Negative Exponents

If a and b are any non-zero real numbers, then $a^{-n} = \frac{1}{a^n}$ and $\frac{1}{b^{-n}} = b^n$. Note this also implies that $\left(\frac{a}{b}\right)^{-n} = \frac{a^{-n}}{b^{-n}} = \frac{b^n}{a^n} = \left(\frac{b}{a}\right)^n$.
In words, when raising a quantity to a negative power, take the reciprocal of the base and change the sign of the exponent.

Simplify the following. Write your answer using positive exponents:

Ex. 4a 11^{-2}

Ex. 4b $(-3)^{-4}$

Ex. 4c $\frac{1}{5^{-2}}$

Ex. 4d $\frac{5}{(-2)^{-4}}$

Ex. 4e $\frac{7^{-2}}{6^{-3}}$

Ex. 4f $(-3x^2)^{-3}$

Ex. 4g $\left(-\frac{4}{y}\right)^{-3}$

Ex. 4h $\left(\frac{1}{8}\right)^{-2}$

Solution:

a) 11^{-2} (apply the definition of a negative exponent)

$= \frac{1}{11^2}$ (simplify)

$= \frac{1}{121}$.

b) $(-3)^{-4}$ (apply the definition of a negative exponent)

$= \frac{1}{(-3)^4}$ (simplify)

$= \frac{1}{81}$.

c) $\frac{1}{5^{-2}}$ (apply the definition of a negative exponent)

$= 5^2$ (simplify)

$= 25$.

d) $\frac{5}{(-2)^{-4}}$ (apply the definition of a negative exponent)

$= 5 \bullet (-2)^4$ (exponents)

$= 5 \bullet 16$ (multiplication)

$= 80$.

e) $\frac{7^{-2}}{6^{-3}}$ (apply the definition of a negative exponent)

$= \frac{6^3}{7^2}$ (simplify)

$= \frac{216}{49}$.

f) $(-3x^2)^{-3}$ (apply the definition of a negative exponent)

$= \frac{1}{(-3x^2)^3}$ (#4 power of a product rule)

$= \frac{1}{(-3)^3(x^2)^3}$ (#3 power rule)

$= \frac{1}{(-3)^3x^6}$ (simplify)

$= -\frac{1}{27x^6}$.

g) $\left(-\frac{4}{y}\right)^{-3}$ (apply the definition of a negative exponent)

$= \left(-\frac{y}{4}\right)^3$ (#5 power of a quotient rule)

$= -\frac{y^3}{4^3}$ (simplify)

$= -\frac{y^3}{64}$.

h) $\left(\frac{1}{8}\right)^{-2}$ (apply the definition of a negative exponent)
$= (8)^2$ (simplify)
$= 64$

Ex. 5a $5x^{-3}$ Ex. 5b $(5x)^{-3}$

Ex. 5c $\frac{-4x^{-3}y^2}{5a^3b^{-5}}$ Ex. 5d $\frac{(-2)^3a^{-3}b^7c^{-5}}{(-7)^2q^{-4}r^2v^0}$

Solution:

a) $5x^{-3}$ (write over 1)
$= \frac{5x^{-3}}{1}$ (apply the definition of a negative exponent)
$= \frac{5}{x^3}$.

b) $(5x)^{-3}$ (write over 1)
$= \frac{(5x)^{-3}}{1}$ (apply the definition of a negative exponent)
$= \frac{1}{(5x)^3}$ (#4 power of a product rule and simplify)
$= \frac{1}{125x^3}$.

c) If the exponents are already positive, do not move the factors. Only move the factors that have negative exponents:

$\frac{-4x^{-3}y^2}{5a^3b^{-5}}$ (negative ÷ positive is negative)

`
$= -\frac{4x^{-3}y^2}{5a^3b^{-5}}$ (apply the definition of a negative exponent)
$= -\frac{4b^5y^2}{5a^3x^3}$. (Note – 4 is not an exponent, but a number so we do not move it).

d) If the exponents are already positive, do not move the factors. Only move the factors that have negative exponents:

$\frac{(-2)^3a^{-3}b^7c^{-5}}{(-7)^2q^{-4}r^2v^0}$ (simplify)
$= \frac{-8a^{-3}b^7c^{-5}}{49q^{-4}r^2v^0}$ (negative ÷ positive is negative)

$= -\frac{8a^{-3}b^7c^{-5}}{49q^{-4}r^2v^0}$ (apply the definition of a negative exponent)

$= -\frac{8b^7q^4}{49a^3c^5r^2v^0}$

But $v^0 = 1$, so $-\frac{8b^7q^4}{49a^3c^5r^2v^0} = -\frac{8b^7q^4}{49a^3c^5r^2(1)} = -\frac{8b^7q^4}{49a^3c^5r^2}$

Concept #3 Properties of Integral Exponents: A Summary

We can now extend the properties of exponents discussed in sections 5.1 and 5.2 to include integral exponents.

Properties of Integral Exponents

Assume that a and b are non-zero real numbers and m and n are integers.

Property	Example	Notes
The Product Rule 1. $a^m a^n = a^{m+n}$	$x^3x^5 = x^{3+5} = x^8$	$x^3x^5 = (x\bullet x\bullet x)(x\bullet x\bullet x\bullet x\bullet x) = x^8$
The Quotient Rule 2. $\frac{b^m}{b^n} = b^{m-n}$	$\frac{b^6}{b^2} = b^{6-2} = b^4$	$\frac{b^6}{b^2} = \frac{\cancel{b}\bullet\cancel{b}\bullet b\bullet b\bullet b\bullet b}{\cancel{b}\bullet\cancel{b}} = b^4$
The Power Rule 3. $(a^m)^n = a^{m\bullet n}$	$(x^3)^2 = x^{3\bullet 2} = x^6$	$(x^3)^2 = (x\bullet x\bullet x)(x\bullet x\bullet x) = x^6$
Power of a Product Rule 4. $(ab)^n = a^nb^n$	$(ab)^4 = a^4b^4$	$(ab)^4 = (ab)(ab)(ab)(ab) = a^4b^4$
Power of a Quotient Rule 5. $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$	$\left(\frac{a}{b}\right)^4 = \frac{a^4}{b^4}$	$\left(\frac{a}{b}\right)^4 = \left(\frac{a}{b}\right)\left(\frac{a}{b}\right)\left(\frac{a}{b}\right)\left(\frac{a}{b}\right) = \frac{a^4}{b^4}$

Definitions

Assume that a is non-zero real number and n is an integers.

Definition	Example	Notes
Zero Exponents $a^0 = 1$	$(-5)^0 = 1$	Any non-zero real number raised to the 0 power is 1.
Negative Exponents $a^{-n} = \frac{1}{a^n}$	$x^{-7} = \frac{1}{x^7}$	Take the reciprocal of the base and change the sign of the exponent.

Concept #4 Simplifying Expressions with Exponents

Simplify the following. Write your answer using positive exponents:

Ex. 6a $\dfrac{x^2}{h^{-7}v^{-8}h^9}$ Ex. 6b $(-7a^2c^{-4})^{-3}$

Ex. 6c $\left(\dfrac{-5x^{-2}y^3}{xy^4}\right)^{-3}$ Ex. 6d $\dfrac{(-7x^3y^4)^{-2}}{(3x^2y^{-2})^{-1}}$

Solution:

a) $\dfrac{x^2}{h^{-7}v^{-8}h^9}$ (#1 product rule in the denominator)

$= \dfrac{x^2}{h^2v^{-8}}$ (apply the definition of a negative exponent)

$= \dfrac{v^8x^2}{h^2}$.

b) $(-7a^2c^{-4})^{-3}$ (#4 power of a product rule)

$= (-7)^{-3}(a^2)^{-3}(c^{-4})^{-3}$ (#3 power rule)

$= (-7)^{-3}a^{-6}c^{12}$ (apply the definition of a negative exponent)

$= \dfrac{c^{12}}{(-7)^3a^6}$ (simplify)

$= -\dfrac{c^{12}}{343a^6}$

c) $\left(\dfrac{-5x^{-2}y^3}{xy^4}\right)^{-3}$ (apply the definition of a negative exponent inside the parenthesis)

$= \left(\dfrac{-5y^3}{x^2xy^4}\right)^{-3}$ (#1 product rule)

$= \left(\dfrac{-5y^3}{x^3y^4}\right)^{-3}$ (#2 quotient rule inside the parenthesis)

$= \left(\dfrac{-5y^{-1}}{x^3}\right)^{-3}$ (apply the definition of a negative exponent inside the parenthesis)

$= \left(\dfrac{-5}{x^3y}\right)^{-3}$ (apply the definition of a negative exponent)

$= \left(\dfrac{x^3y}{-5}\right)^{3}$ (#5 power of a quotient rule)

$= \frac{(x^3y)^3}{(-5)^3}$ (#4 power of a product rule)

$= \frac{(x^3)^3(y)^3}{(-5)^3}$ (#3 power rule)

$= \frac{x^9y^3}{(-5)^3}$ (simplify)

$= -\frac{x^9y^3}{125}$

d) $\frac{(-7x^3y^4)^{-2}}{(3x^2y^{-2})^{-1}}$ (#4 power of a product rule)

$= \frac{(-7)^{-2}(x^3)^{-2}(y^4)^{-2}}{(3)^{-1}(x^2)^{-1}(y^{-2})^{-1}}$ (#3 power rule)

$= \frac{(-7)^{-2}x^{-6}y^{-8}}{(3)^{-1}x^{-2}y^2}$ (apply the definition of a negative exponent)

$= \frac{3x^2}{(-7)^2x^6y^8y^2}$ (#1 product rule)

$= \frac{3x^2}{(-7)^2x^6y^{10}}$ (#2 quotient rule)

$= \frac{3x^{-4}}{(-7)^2y^{10}}$ (apply the definition of a negative exponent)

$= \frac{3}{(-7)^2x^4y^{10}}$ (simplify)

$= \frac{3}{49x^4y^{10}}$

Ex. 7 $3^{-1} + \left(\frac{7}{2}\right)^{-2} - 7^0$

<u>Solution:</u>

$3^{-1} + \left(\frac{7}{2}\right)^{-2} - 7^0$ (apply the definition of a negative exponent)

$= \frac{1}{3} + \left(\frac{2}{7}\right)^2 - 7^0$ (#5 quotient to a power and simplify)

$= \frac{1}{3} + \frac{4}{49} - 1$ (LCD = 147 and simplify)

$= \frac{49}{147} + \frac{12}{147} - \frac{147}{147} = -\frac{86}{147}$