# Sect 5.6 - Multiplying Polynomials

Concept #1 Multiplication of Polynomials

**Multiplying a monomial and a polynomial**
Recall in section 1.6 how we used the distributive property to simplify the following example:

**Simplify:**

Ex. 1 $-4(-3x + 6 - 2y)$

Solution:

$\mathbf{-4}(-3x + 6 - 2y) = \mathbf{-4}(-3x) + (\mathbf{-4})\bullet 6 - (\mathbf{-4})\bullet 2y$

$= 12x + (-24) + 8y = 12x - 24 + 8y.$

We can extend this idea to multiplying a polynomial by a monomial. Suppose in example 1, we change the $-4$ to $-4x^2y$. Let's see what happens:

Ex. 2 $-4x^2y(-3x + 6 - 2y)$

Solution:

$\mathbf{-4x^2y}(-3x + 6 - 2y) = \mathbf{-4x^2y}(-3x) + (\mathbf{-4x^2y})\bullet 6 - (\mathbf{-4x^2y})\bullet 2y$

$= 12x^3y + (-24x^2y) + 8x^2y^2 = 12x^3y - 24x^2y + 8x^2y^2.$

Ex. 3 $-\frac{2}{3}a^2\left(5a^3 - 6a^2 + \frac{1}{4}a - \frac{2}{7}\right)$

Solution:

$-\frac{2}{3}\mathbf{a^2}\left(5a^3 - 6a^2 + \frac{1}{4}a - \frac{2}{7}\right)$

$= -\frac{2}{3}\mathbf{a^2}\left(\frac{5}{1}a^3\right) - \left(-\frac{2}{3}\mathbf{a^2}\right)\frac{6}{1}a^2 + \left(-\frac{2}{3}\mathbf{a^2}\right)\left(\frac{1}{4}a\right) - \left(-\frac{2}{3}\mathbf{a^2}\right)\left(\frac{2}{7}\right)$

$= -\frac{2}{3}a^2\left(\frac{5}{1}a^3\right) - \left(-\frac{2}{1}a^2\right)\frac{2}{1}a^2 + \left(-\frac{1}{3}a^2\right)\left(\frac{1}{2}a\right) - \left(-\frac{2}{3}a^2\right)\left(\frac{2}{7}\right)$

$= -\frac{10}{3}a^5 + 4a^4 - \frac{1}{6}a^3 + \frac{4}{21}a^2.$

Ex. 4 $-0.7cd(3c^2 - 4cd + 2d^2)$

Solution:

$\mathbf{-0.7cd}(3c^2 - 4cd + 2d^2)$

$= \mathbf{-0.7cd}(3c^2) - (\mathbf{-0.7cd})(4cd) + (\mathbf{-0.7cd})(2d^2)$

$= -2.1c^3d + 2.8c^2d^2 - 1.4cd^3$

Ex. 5 $Q(3a + 5)$

Solution:

$\mathbf{Q}(3a + 5) = \mathbf{Q}(3a) + Q(5) = 3a\mathbf{Q} + 5\mathbf{Q}$.

**Multiplying a polynomial by a polynomial horizontally.**

In example five, let's suppose that we changed Q to (2a – 3). To simplify the problem, we would still distribute the (2a – 3) in the same fashion that we distributed the Q in example five. Afterwards, we would have use the distributive property again to finish the problem. Let's have a look:

**Simplify:**

Ex. 6 $(2a - 3)(3a + 5)$

Solution:

$\mathbf{(2a - 3)}(3a + 5) = \mathbf{(2a - 3)}(3a) + (2a - 3)(5) = 3a\mathbf{(2a - 3)} + 5\mathbf{(2a - 3)}$

Now, distribute the 3a and the 5:

$3a\mathbf{(2a - 3)} + 5\mathbf{(2a - 3)} = 3a\mathbf{(2a)} - 3a\mathbf{(3)} + 5\mathbf{(2a)} - 5\mathbf{(3)}$

$= 6a^2 - 9a + 10a - 15$. Finally, combine like terms:

$= 6a^2 + a - 15$.

Ex. 7 $(3x - 0.4y)(2x^2 - 7xy + 1.3y^2)$

Solution:

$\mathbf{(3x - 0.4y)}(2x^2 - 7xy + 1.3y^2)$

$= 2x^2\mathbf{(3x - 0.4y)} - 7xy\mathbf{(3x - 0.4y)} + 1.3y^2\mathbf{(3x - 0.4y)}$

$= 2x^2\mathbf{(3x)} - 2x^2\mathbf{(0.4y)} - 7xy\mathbf{(3x)} - (-7xy)\mathbf{0.4y} + 1.3y^2\mathbf{(3x)} - 1.3y^2\mathbf{(0.4y)}$

$= 6x^3 - 0.8x^2y - 21x^2y + 2.8xy^2 + 3.9xy^2 - 0.52y^3$

$= 6x^3 - 21.8x^2y + 6.7xy^2 - 0.52y^3$

Ex. 8 $(3x + 2y)^2$

Solution:

$(3x + 2y)^2 = \mathbf{(3x + 2y)}(3x + 2y)$

$= 3x\mathbf{(3x + 2y)} + 2y\mathbf{(3x + 2y)}$

$= 3x\mathbf{(3x)} + 3x\mathbf{(2y)} + 2y\mathbf{(3x)} + 2y\mathbf{(2y)} = 9x^2 + 6xy + 6xy + 4y^2$

$= 9x^2 + 12xy + 4y^2$.

Multiplying a polynomial by a polynomial vertically.

When multiplying two numbers in arithmetic like 684 and 73, we do not multiply them horizontally, but multiply them vertically taking care to line up the place values. Let's look at an example:

**Simplify:**

Ex. 9 $684 \times 73$

Solution:

```
 21        4 × 3 = 12, write down 2, carry the 1.
 684       8 × 3 = 24, 24 + 1 = 25, write down 5, carry the 2.
×  3       6 × 3 = 18, 18 + 2 = 20, write down 20
2052  (1st partial product)
           Write down a zero.
 52        4 × 7 = 28, write down 8, carry the 2.
 684       8 × 7 = 56, 56 + 2 = 58, write down 8, carry the 5.
× 70       6 × 7 = 42, 42 + 5 = 47, write down 47
47880  (2nd partial product)
```

Now add the two partial products:

```
    684
  × 73
   1
   2052   ← 684 × 3
+ 47880   ← 684 × 70
 49,932
```

So, the answer is 49,932.

We can do the same thing with polynomial except instead of aligning place values, we align like terms and we do not have any carries:

Ex. 10 $(6x^2 + 8x - 4)(7x - 3)$

Solution:

Start with the – 3 and multiply each term of the trinomial by – 3:

$-4 \bullet \mathbf{-3} = 12$, write down + 12.

$6x^2 + 8x - 4$ $\quad$ $8x \bullet \mathbf{-3} = -24x$, write down $-24x$.

$\underline{\qquad\qquad\mathbf{-3}}$ $\quad$ $6x^2 \bullet \mathbf{-3} = -18x^2$, write down $-18x^2$.

$-18x^2 - 24x + 12$ $\quad$ (1st partial product)

Now, move to the 7x and multiply each term of the trinomial by 7x:

$-4 \bullet \mathbf{7x} = -28x$, write down $-28x$.

$6x^2 + 8x - 4$ $\quad$ $8x \bullet \mathbf{7x} = 56x^2$, write down $+56x^2$.

$\underline{\qquad\qquad\mathbf{7x}}$ $\quad$ $6x^2 \bullet \mathbf{7x} = 42x^3$, write down $+42x^3$.

$42x^3 + 56x^2 - 28x$ $\quad$ (2nd partial product)

Now add by combine like terms:

$$\begin{array}{r} 6x^2 + 8x - 4 \\ \underline{\mathbf{7x - 3}} \\ -18x^2 - 24x + 12 \\ +42x^3 + 56x^2 - 28x \phantom{+12} \\ \hline 42x^3 + 38x^2 - 52x + 12 \end{array}$$

Ex. 11 $(3x^2 - x + 2)(-8x^2 + 2x - 5)$

<u>Solution:</u>

Start with the – 5 and multiply each term of $(3x^2 - x + 2)$ by – 5:

$$\begin{array}{r} 3x^2 - x + 2 \\ \mathbf{-5} \\ \hline -15x^2 + 5x - 10 \end{array}$$

$2 \bullet \mathbf{-5} = -10.$
$-x \bullet \mathbf{-5} = +5x.$
$3x^2 \bullet \mathbf{-5} = -15x^2.$
(1st partial product)

Now, move to the 2x and multiply each term of $(3x^2 - x + 2)$ by 2x:

$$\begin{array}{r} 3x^2 - x + 2 \\ \mathbf{2x} \\ \hline 6x^3 - 2x^2 + 4x \phantom{+2} \end{array}$$

$2 \bullet \mathbf{2x} = +4x.$
$-x \bullet \mathbf{2x} = -2x^2.$
$3x^2 \bullet \mathbf{2x} = +6x^3.$
(2nd partial product)

Finally, multiply each term of $(3x^2 - x + 2)$ by $-8x^2$:

$$\begin{array}{r} 3x^2 - x + 2 \\ \mathbf{-8x^2} \\ \hline -24x^4 + 8x^3 - 16x^2 \phantom{-x+2} \end{array}$$

$2 \bullet \mathbf{-8x^2} = -16x^2.$
$-x \bullet \mathbf{-8x^2} = +8x^3.$
$3x^2 \bullet \mathbf{-8x^2} = -24x^4.$
(3rd partial product)

Now add by combine like terms:

$$\begin{array}{r} 3x^2 - x + 2 \\ -8x^2 + 2x \mathbf{-5} \\ \hline -15x^2 + 5x - 10 \\ 6x^3 - 2x^2 + 4x \phantom{-10} \\ -24x^4 + 8x^3 - 16x^2 \phantom{+4x-10} \\ \hline -24x^4 + 14x^3 - 33x^2 + 9x - 10 \end{array}$$

Ex. 12 $(2x - 7)(3x - 4)(3x + 4)$

Solution:

First, multiply two of the polynomials. Then multiply the answer with the third polynomial:

$\mathbf{(3x - 4)}(3x + 4)$

$= 3x\mathbf{(3x - 4)} + 4\mathbf{(3x - 4)}$

$= 3x\mathbf{(3x)} - 3x\mathbf{(4)} + 4\mathbf{(3x)} - (4)\mathbf{(4)}$

$= 9x^2 - 12x + 12x - 16$

$= 9x^2 - 16.$

So, $(2x - 7)\mathbf{(3x - 4)(3x + 4)} = (2x - 7)\mathbf{(9x^2 - 16)}$

$$
\begin{array}{r}
2x - 7 \\
\underline{9x^2 \qquad - 16} \\
-32x + 112 \\
\underline{18x^3 - 63x^2 \qquad\qquad} \\
18x^3 - 63x^2 - 32x + 112
\end{array}
$$

Concept #2 Special Case Products

F.O.I.L. is a short cut for multiplying two binomials. Suppose we are multiplying $(3x - 5)(2x + 7)$. The first terms of the binomials are 3x and 2x. **F.** stands for the product of the first terms.

$(\mathbf{3x} - 5)(\mathbf{2x} + 7)$ **F.** $\rightarrow (3x)(2x) = 6x^2$

The outer terms of the binomials are next to the parentheses on the ends. They are 3x and 7. **O.** stands for the product of the outer terms.

$(\mathbf{3x} - 5)(2x \mathbf{+ 7})$ **O.** $\rightarrow (3x)(7) = 21x$

The inner terms of the binomials are next to the parentheses in the middle. They are – 5 and 2x. **I.** stands for the product of the inner terms.

$(3x \mathbf{- 5})(\mathbf{2x} + 7)$ **I.** $\rightarrow (-5)(2x) = -10x$

The last terms of the binomials are – 5 and 7. **L.** stands for the product of the last terms.

$(3x \mathbf{- 5})(2x \mathbf{+ 7})$ **L.** $\rightarrow (-5)(7) = -35$

**F. O. I. L.**

So, $(\mathbf{3x} - 5)(\mathbf{2x} + 7) = 6x^2 + 21x - 10x - 35 = 6x^2 + 11x - 35.$

**<u>Simplify using FOIL:</u>**

Ex. 13 $(7x^2 - 5y)(3x + 8y)$

Solution:

**F. O. I. L.**

$(7x^2 - 5y)(3x + 8y) = (7x^2)(3x) + (7x^2)(8y) + (-5y)(3x) + (-5y)(8y)$

$= 21x^3 + 56x^2y - 15xy - 40y^2$

Ex. 14 (0.3a – 0.4b)(0.7a + 0.9b)

Solution:

$$(0.3a - 0.4b)(0.7a + 0.9b) = \underset{\textbf{F.}}{0.21a^2} + \underset{\textbf{O.}}{0.27ab} - \underset{\textbf{I.}}{0.28ab} - \underset{\textbf{L.}}{0.36b^2}$$
$$= 0.21a^2 - 0.01ab - 0.36b^2.$$

Ex. 15 $\left(\frac{4}{9}x - \frac{4}{7}\right)\left(\frac{5}{9}x - \frac{8}{7}\right)$

Solution:

$$= \underset{\textbf{F.}}{\left(\frac{4}{9}x\right)\left(\frac{5}{9}x\right)} + \underset{\textbf{O.}}{\left(\frac{4}{9}x\right)\left(-\frac{8}{7}\right)} + \underset{\textbf{I.}}{\left(-\frac{4}{7}\right)\left(\frac{5}{9}x\right)} + \underset{\textbf{L.}}{\left(-\frac{4}{7}\right)\left(-\frac{8}{7}\right)}$$
$$= \frac{20}{81}x^2 - \frac{32}{63}x - \frac{20}{63}x + \frac{32}{49} = \frac{20}{81}x^2 - \frac{52}{63}x + \frac{32}{49}.$$

To see where the formulas for the Perfect Square Trinomials come from, let's work the following examples:

**<u>Simplify:</u>**

Ex. 16 $(F + L)^2$

Solution:

**(F + L)**(F + L) = **(F + L)**(F) + **(F + L)**(L) = F**(F + L)** + L**(F + L)**

Now, distribute the F and the L:

F**(F + L)** + L**(F + L)** = F**(F) +** F**(L)** + L**(F) +** L**(L)**

$= F^2 + FL + FL + L^2$. Finally, combine like terms:

$= F^2 + 2FL + L^2$.

Ex. 17 $(F - L)^2$

Solution:

**(F – L)**(F – L) = **(F – L)**(F) – **(F – L)**(L) = F**(F – L)** – L**(F – L)**

Now, distribute the F and the – L:

F**(F – L)** – L**(F – L)** = F**(F) –** F**(L)** – L**(F) –** (– L)**(L)**

$= F^2 - FL - FL + L^2$. Finally, combine like terms:

$= F^2 - 2FL + L^2$.

**<u>The Perfect Square Trinomials:</u>**

1) $(F + L)^2 = F^2 + 2FL + L^2$
2) $(F - L)^2 = F^2 - 2FL + L^2$

**<u>Use a perfect square trinomial to square each binomial:</u>**

Ex. 18 $(0.8x + 0.5)^2$

Solution:

According to our special product formula $(F + L)^2 = F^2 + 2FL + L^2$, since F = 0.8x and L = 0.5, the answer should be:

$(F + L)^2 = F^2 + 2FL + L^2$

$([0.8x] + [0.5])^2 = [0.8x]^2 + 2[0.8x][0.5] + [0.5]^2$

$= 0.64x^2 + 0.8x + 0.25$.

Ex. 19 $(2x – y)^2$

Solution:

According to our special product formula $(F – L)^2 = F^2 – 2FL + L^2$, since F = 2x and L = y, the answer should be:

$(F – L)^2 = F^2 – 2FL + L^2$

$([2x] – [y])^2 = [2x]^2 – 2[2x][y] + [y]^2 = 4x^2 – 4xy + y^2$.

Ex. 20 $\left(\frac{5}{6}x - \frac{8}{7}y\right)^2$

Solution:

According to our special product formula $(F – L)^2 = F^2 – 2FL + L^2$, since $F = \frac{5}{6}x$ and $L = \frac{8}{7}y$, the answer should be:

$(F – L)^2 = F^2 – 2FL + L^2$

$\left(\left[\frac{5}{6}x\right] - \left[\frac{8}{7}y\right]\right)^2 = \left[\frac{5}{6}x\right]^2 - 2\left[\frac{5}{6}x\right]\left[\frac{8}{7}y\right] + \left[\frac{8}{7}y\right]^2$

$= \frac{25}{36}x^2 - \frac{2}{1}\left[\frac{5}{6}x\right]\left[\frac{8}{7}y\right] + \frac{64}{49}y^2 = \frac{25}{36}x^2 - \frac{1}{1}\left[\frac{5}{3}x\right]\left[\frac{8}{7}y\right] + \frac{64}{49}y^2$

$= \frac{25}{36}x^2 - \frac{40}{21}xy + \frac{64}{49}y^2$.

We see that these special products work for these special cases. It also shows that $(x + y)^2 \neq x^2 + y^2$ and $(x – y)^2 \neq x^2 – y^2$. When there is more than one term inside the parentheses, we need to write the problem in expanded form and then use are techniques for multiplying polynomials. To see where the formulas for the Difference of Squares come from, let’s work the following example:

**Simplify:**

Ex. 21 (F + L)(F – L)

Solution:

**(F + L)**(F – L) = **(F + L)**(F) – **(F + L)**(L) = F**(F + L)** – L**(F + L)**

Now, distribute the F and the – L:

F**(F + L)** – L**(F + L)** = F**(F)** + F**(L)** – L**(F)** + – L**(L)**

$= F^2 + FL - FL - L^2$. Finally, combine like terms:

$= F^2 - L^2$.

**The Difference of Squares**

$(F - L)(F + L) = F^2 - L^2$

Note (F + L) and (F – L) are called conjugates

**Use the difference of squares to simplify.**

Ex. 22 (2.3x – 1.6)(2.3x + 1.6)

Solution:

According to our special product formula $(F - L)(F + L) = F^2 - L^2$, since F = 2.3x and L = 1.6, the answer should be:

$(F - L)(F + L) = F^2 - L^2$

$([2.3x] - [1.6])([2.3x] + [1.6]) = [2.3x]^2 - [1.6]^2 = 5.29x^2 - 2.56.$

Ex. 23 $\left(\frac{4}{9}x + \frac{2}{13}y\right)\left(\frac{4}{9}x - \frac{2}{13}y\right)$

Solution:

According to our special product formula $(F - L)(F + L) = F^2 - L^2$, since $F = \frac{4}{9}x$ and $L = \frac{2}{13}y$, the answer should be:

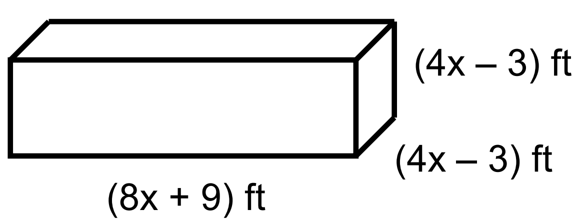
$(F - L)(F + L) = F^2 - L^2$

$$\left(\left[\frac{4}{9}x\right] - \left[\frac{2}{13}y\right]\right)\left(\left[\frac{4}{9}x\right] + \left[\frac{2}{13}y\right]\right) = \left[\frac{4}{9}x\right]^2 - \left[\frac{2}{13}y\right]^2 = \frac{16}{81}x^2 - \frac{4}{169}y^2.$$

Concept #3 Applications to Geometry

**Find the volume of the following:**

Ex. 24



<u>Solution:</u>

$V = Lwh = (8x + 9)(4x – 3)(4x – 3)$

First, multiply two of the polynomials. Then multiply the answer with the third polynomial:

$(4x – 3)(4x – 3)$

F. O. I. L

$= 4x(4x) – 4x(3) – 3(4x) – (– 3)(3) = 16x^2 – 12x – 12x + 9$

$= 16x^2 – 24x + 9.$

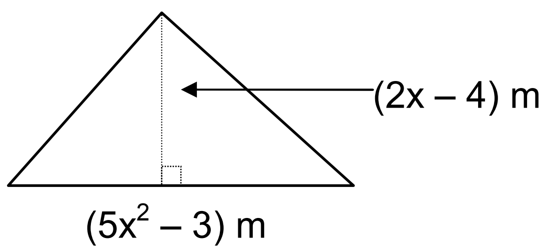
So, $(8x + 9)\mathbf{(4x – 3)(4x – 3)} = (8x + 9)\mathbf{(16x^2 – 24x + 9)}$

$$\begin{array}{r} 16x^2 – 24x + 9 \\ \underline{8x + 9} \\ 144x^2 – 216x + 81 \\ \underline{128x^3 – 192x^2 + 72x \qquad\qquad} \\ 128x^3 – 48x^2 – 144x + 81 \qquad \end{array}$$

So, $V = (128x^3 – 48x^2 – 144x + 81)\ ft^3$.

**<u>Find the area of the triangle:</u>**

Ex. 25




<u>Solution:</u>

The formula for the area of a triangle is $A = \frac{1}{2}bh$. The base $b = (5x^2 – 3)$ m and the height $h = (2x – 4)$ m. Thus,

$A = \frac{1}{2}bh = \frac{1}{2}(5x^2 – 3)(2x – 4)$ (F.O.I.L.)

F. O. I. L.

$A = \frac{1}{2}[(5x^2)(2x) + (5x^2)(– 4) – 3(2x) – 3(– 4)]$ (simplify)

$A = \frac{1}{2}[10x^3 – 20x^2 – 6x + 12]$ (distribute the $\frac{1}{2}$)

$A = 5x^3 – 10x^2 – 3x + 6$

Thus, the area is $(5x^3 – 10x^2 – 3x + 6)\ m^2$.